ON THE RELATION BETWEEN AN OPERATOR AND ITS SELF-COMMUTATOR

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Let X and Y be bounded self-adjoint operators on a Hilbert space H. The paper deals with the following well known problem: if the commutator $[X,Y]$ is small in an appropriate sense, is there a pair of commuting operators $\tilde{X}$ and $\tilde{Y}$ which are close to X and Y? Note that for general bounded operators X and Y it is not necessarily true (see Subsection B.8).

For self-adjoint X and Y, taking $A := X + iY$, one can reformulate the question as follows: if the self-commutator $[A^*, A]$ is small, is there a normal operator $\widetilde{A}$ close to A? There are some positive results in this direction. Probably, the most famous is the Brown–Douglas–Fillmore theorem [BDF].

Theorem 0.1. *If H is separable, $[A^*, A]$ is compact and the corresponding to A element of the Calkin algebra has trivial index function then there is a normal operator T such that $A - T$ is compact.*

Another is due to Huaxin Lin [L3].

Theorem 0.2. *There exists a continuous function $F : [0,\infty) \mapsto [0,1]$ vanishing at the origin such that the distance from A to the set of normal operators is estimated by $F(\|[A^*, A]\|)$ for all finite rank operators A with $\|A\| \leqslant 1$.*

A related question is whether an operator A with small self-commutator is close to a diagonal operator. Recall that an operator T on a separable Hilbert space is said to be diagonal if it is represented by a diagonal matrix in some orthonormal basis. Clearly, all diagonal operators are normal. The following result was obtained in [Be] and is usually referred to as the Weyl–von Neumann–Berg theorem.

Theorem 0.3. *Let A be a (not necessarily bounded) normal operator on a separable Hilbert space. Then for each $\varepsilon > 0$ there exist a diagonal operator D_ε and a compact operator K_ε with $\|K_\varepsilon\| \leqslant \varepsilon$ such that $A = D_\varepsilon + K_\varepsilon$.*

Our main result is Theorem 2.12, which shows that a bounded operator A belongs to a certain set associated with its self-commutator whenever $A - \lambda I$ can be approximated by invertible operators for all $\lambda \in \mathbb{C}$. Theorem 2.12 implies both the BDF and Huaxin Lin's theorems. Moreover, it allows us to refine the former and to extend the latter to operators of infinite rank and other norms (see Subsections 3.1 and 3.3, Corollary 2.14 and Remark 2.11). In particular, we obtain

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(i) a quantitative version of Theorem 0.1, which links the BDF and Weyl–von Neumann–Berg theorems for bounded operators, and
(ii) an analogue of Huaxin Lin's theorem for the Schatten norms of finite matrices.

Theorem 2.12 holds for any unital C^*-algebra L of real rank zero, but in the general case we need a slightly stronger condition on A. Namely, we assume that $A-\lambda I$ belong to the closure of the connected component of unity in the set of invertible elements in L for all $\lambda\in\mathbb{C}$.

Our proof of Theorem 2.12 uses the C^*-algebra technique developed in [FR1] and [FR2] and cannot be significantly simplified by assuming that L is the C^*-algebra of all bounded operators. One of the main ingredients in the proof is the part of Corollary 2.5 which says that a normal operator satisfying the above condition can be approximated by normal operators with finite spectra. This statement is contained in [FR2, Theorem 3.2]. The authors indicated how it could be proved but did not present complete arguments. Therefore, for reader's convenience, we give operator-theoretic proofs of this and relevant results. More precisely, we deduce Corollary 2.5 from Theorem 2.1, which seems to be new and may be of independent interest.

1. Notation and auxiliary results

1.1. **Notation and definitions.** Let H be a complex Hilbert space (not necessarily separable). Throughout the paper,

- $\mathcal{B}(H)$ is the C^*-algebra of all bounded operators in H;
- $\sigma(A)$ denotes the spectrum of $A\in\mathcal{B}(H)$;
- L is a unital C^*-algebra represented on the Hilbert space H, so that $L\subseteq\mathcal{B}(H)$.

Recall that, by the Gelfand–Naimark theorem, such a representation exists for any unital C^*-algebra L.

Every operator $A\in\mathcal{B}(H)$ admits the polar decomposition $A=V|A|$, where $|A|$ is the self-adjoint operator $\sqrt{A^*A}$ and V is an isometric operator such that $VH=\overline{AH}$ and $\ker V=\ker A=\ker|A|$. If A is normal then $V|A|=|A|V$.

Remark 1.1. If $A\in L$ then $f(|A|)\in L$ for any continuous function f. If, in addition, A is invertible then $V=A|A|^{-1}$ is a unitary element of L because $|A|^{-1}$ can be approximated by continuous functions of $|A|$. In particular, this implies that $A^{-1}\in L$. If $A\in L$ is not invertible then the isometric operator V in its polar representation does not have to belong to L.

Remark 1.2. The spectral projections of a normal operator $A\in L$ may not lie in L. However, the spectral projection corresponding to a connected component of $\sigma(A)$ belongs to L, since it can be written as a continuous function of A.

Further on

- L^{-1} is the set of invertible operators in L;
- L^{-1}_0 denotes the connected component of L^{-1} containing the identity operator;
- L_{n}, L_{u}, and L_{s} are the sets of normal, unitary and self-adjoint operators in L respectively;

- L_{f} is the set of operators $A \in L$ with finite spectra;
- if $M \subset L$ then $\overline{M}$ denotes the norm closure of the set M in L.

Clearly, the sets L^{-1} and L_0^{-1} are open in L, and the sets L_{n}, L_{u} and L_{s} are closed.

One says that

- L has *real rank zero* if $\overline{L^{-1} \cap L_{\mathrm{s}}} = L_{\mathrm{s}}$.

The concept of real rank of a C^*-algebra was introduced in [BP]. A unital C^*-algebra L has real rank zero if and only if any self-adjoint operator $A \in L$ is the norm limit of a sequence of self-adjoint operators from L with finite spectra (see Corollary 2.4 and Subsection B.1).

Remark 1.3. Note that any self-adjoint operator $A \in \mathcal{B}(H)$ is approximated in the norm topology by invertible self-adjoint operators of the form $f(A)$, where f are suitable real-valued Borel functions. Therefore, all von Neumann algebras (in particular, $\mathcal{B}(H)$ and the algebra of finite $m \times m$-matrices) have real rank zero.

Example 1.4. The minimal unital C^*-algebra L_A containing a given bounded self-adjoint operator A consists of normal operators $f(A)$, where f are continuous complex-valued functions on $\sigma(A)$. If there is an open interval $(a,b) \subset \sigma(A)$ then $A - \frac{a+b}{2} I \notin \overline{L^{-1} \cap L_{\mathrm{s}}}$ and, consequently, L_A is not of real rank zero.

Our main results hold for C^*-algebras of real rank zero and operators $A \in L$ satisfying the following condition

(C) $A - \lambda I \in \overline{L_0^{-1}}$ for all $\lambda \in \mathbb{C}$.

1.2. **Auxiliary lemmas.** We shall need the following simple lemmas.

Lemma 1.5. *Let $\widehat{L}$ be the subset of the direct product $\mathbb{C} \times L$ which consists of all pairs (λ, A) such that $\lambda \notin \sigma(A)$. If $A_0 - \lambda_0 I \in L_0^{-1}$ for some $(\lambda_0, A_0) \in \widehat{L}$ then $A - \lambda I \in \overline{L_0^{-1}}$ for all (λ, A) lying in the closure of the connected component of $\widehat{L}$ that contains (λ_0, A_0).*

Proof. Let $\widehat{L}_0$ be the connected component of $\widehat{L}$ containing (λ_0, A_0). Since the set $\widehat{L}$ is open, $\widehat{L}_0$ is path-connected. If $(\lambda, A) \in \widehat{L}_0$ and $(\lambda_t, A_t) \subset \widehat{L}_0$ is a path in $\widehat{L}_0$ from (λ_0, A_0) to (λ, A) then $A_t - \lambda_t I \in L^{-1}$ for all t, which implies that $A - \lambda I \in L_0^{-1}$. If (λ, A) belongs to the closure of $\widehat{L}_0$ then $A - \lambda I$ can be approximated by operators $A_n - \lambda_n I \in L_0^{-1}$ with $(\lambda_n, A_n) \in \widehat{L}_0$. □

Remark 1.6. If $|\mu| > \|A\|$ then $A - \mu I \in L_0^{-1}$ because $[0,1] \ni t \mapsto tA - \mu I$ is a path in L^{-1} from $-\mu I \in L_0^{-1}$ to $A - \mu I$. Therefore, Lemma 1.5 implies that A satisfies the condition **(C)** whenever $\mathbb{C} \setminus \sigma(A)$ is a dense connected subset of $\mathbb{C}$. In particular, **(C)** is fulfilled for all compact operators $A \in L$, all $A \in L_{\mathrm{s}} \cup L_{\mathrm{f}}$, and all unitary operators $A \in L_{\mathrm{u}}$ whose spectra do not coincide with the whole unit circle.

Lemma 1.7. *Let $A \in L^{-1}$ and $A = U|A|$. Then $A \in L_0^{-1}$ if and only if $U \in L_0^{-1} \cap L_{\mathrm{u}}$.*

Proof. $[0,1] \ni t \mapsto U(tI + (1-t)|A|)$ is a path in L^{-1} from A to U. □

Lemma 1.8. *Assume that L has real rank zero. Then $U \in L_{\mathrm{u}} \cap L_0^{-1}$ if and only if for every $\varepsilon \in (0,1)$ there exist unitary operators $U_\varepsilon, W_\varepsilon \in L_{\mathrm{u}}$ such that $U = W_\varepsilon U_\varepsilon$, $-1 \notin \sigma(U_\varepsilon)$ and $\|W_\varepsilon - I\| \leqslant \varepsilon$.*

Proof. Recall that the point -1 does not belong to the spectrum of $U \in L_{\mathrm{u}}$ if and only if U is the Cayley transform of a self-adjoint operator X, that is, $U = (iI - X)^{-1}(iI + X)$ where $X = i\,(U + I)^{-1}(U - I) \in L_{\mathrm{s}}$. For every such an operator U, the principal branch of the argument Arg is continuous in a neighbourhood of $\sigma(U)$, so that $\operatorname{Arg} U \in L_{\mathrm{s}}$ and $\exp(it \operatorname{Arg} U)$ is a path in $L_{\mathrm{u}} \cap L_0^{-1}$ from I to U.

Assume first that $U = W_\varepsilon U_\varepsilon$ where U_ε and W_ε satisfy the above conditions. Then $U \in L_{\mathrm{u}}$, $-1 \notin \sigma(W_\varepsilon)$ and $\exp(it \operatorname{Arg} W_\varepsilon) \exp(it \operatorname{Arg} U_\varepsilon)$ is a path in $L_{\mathrm{u}} \cap L_0^{-1}$ from I to U. Thus $U \in L_{\mathrm{u}} \cap L_0^{-1}$.

Assume now that $U \in L_{\mathrm{u}} \cap L_0^{-1}$. Then there exists a path $Z(t) \subset L_0^{-1}$ from I to U. The "normalized" path $\tilde Z(t) = Z(t)|Z(t)|^{-1}$ lies in $L_{\mathrm{u}} \cap L_0^{-1}$ and also joins I and U. Let us choose a finite collection of points $0 = t_0 < t_1 < t_2 < \cdots < t_m = 1$ such that $\|\tilde Z(t_j) - \tilde Z(t_{j-1})\| < 1$ and define $V_j := \tilde Z(t_j)\,\tilde Z^{-1}(t_{j-1})$. Then $U = V_m V_{m-1} \ldots V_1$ and $\|V_j - I\| < 1$, so that $-1 \notin \sigma(V_j)$ for all j.

Let $V = (iI - Y)^{-1}(iI + Y)$ and $\tilde V = (iI - \tilde Y)^{-1}(iI + \tilde Y)$ where $Y, \tilde Y \in L_{\mathrm{s}}$. Since L has real rank zero, for each $\delta > 0$ we can find $Y_\delta \in L_{\mathrm{s}} \cap L^{-1}$ such that $\|Y - Y_\delta\| < \delta$, and then $\tilde Y_\delta \in L_{\mathrm{s}}$ such that $\|\tilde Y - \tilde Y_\delta\| < \delta$ and $\tilde Y_\delta - Y_\delta^{-1} \in L^{-1}$. If $V_\delta = (iI - Y_\delta)^{-1}(iI + Y_\delta)$ and $\tilde V_\delta = (iI - \tilde Y_\delta)^{-1}(iI + \tilde Y_\delta)$ then $\|V_\delta \tilde V_\delta - V \tilde V\| \to 0$ as $\delta \to 0$ because the function $t \mapsto (iI - t)^{-1}(iI + t)$ is continuous, and $-1 \notin \sigma(V_\delta \tilde V_\delta)$ because

$$
\begin{aligned}
(iI - Y_\delta)\,(V_\delta \tilde V_\delta + I)\,(iI - \tilde Y_\delta) \;&=\; (iI + Y_\delta)\,(iI + \tilde Y_\delta) + (iI - Y_\delta)(iI - \tilde Y_\delta) \\
&=\; 2\,(Y_\delta \tilde Y_\delta - I) = 2\,Y_\delta\,(\tilde Y_\delta - Y_\delta^{-1}) \;\in\; L^{-1}\,.
\end{aligned}
$$

Thus we see that the composition of two unitary operators whose spectra do not contain -1 can be approximated by unitary operators with the same property. By induction, the same is true for the composition of any finite collection of unitary operators. In particular, there exists $U_\varepsilon \in L_{\mathrm{u}} \cap L_0^{-1}$ such that $-1 \notin \sigma(U_\varepsilon)$ and $\|U - U_\varepsilon\| < \varepsilon$. Taking $W_\varepsilon := U\,U_\varepsilon^{-1}$, we obtain the required representation of U. $\square$

Lemma 1.9. *Assume that L has real rank zero. Then*

(1) *for every $A \in \overline{L_0^{-1}}$ and every $\delta > 0$ there exists an operator $S_\delta \in L_0^{-1}$ such that $\|S_\delta^{-1}\| \leqslant \delta^{-1}$ and $\|A - S_\delta\| \leqslant 2\delta$.*

(2) *If $A \in L_{\mathrm{s}}$ then one can find a self-adjoint operator $S_\delta \in L^{-1}$ satisfying the above conditions.*

(3) *For every $S \in L_0^{-1}$ there exists a continuous path $Z : [0,1] \mapsto L_0^{-1}$ such that $Z(0) = I$, $Z(1) = S$, $\|Z(t)^{-1}\| \leqslant \max\{1\,,\,\|S^{-1}\|\}$ for all $t \in [0,1]$ and*

$$
\|Z(t) - Z(r)\| \;\leqslant\; |t - r|\,(1 + 2\pi)\,\max\{1\,,\,\|S\|\}\,, \qquad \forall t, r \in [0,1]\,. \tag{1.1}
$$

Proof.

(1) Let us choose an arbitrary operator $B \in L_0^{-1}$ such that $\|A - B\| \leqslant \delta$, and let $B = V|B|$ be its polar decomposition. Then, by Lemma 1.7, we have $V \in L_0^{-1} \cap L_{\mathrm{u}}$ and $S_\delta := V((|B| - \delta I)_+ + \delta I) \in L_0^{-1}$. Obviously, $\|S_\delta^{-1}\| = \|((|B| - \delta I)_+ + \delta I)^{-1}\| \leqslant \delta^{-1}$ and $\|B - S_\delta\| = \|(|B| - \delta I)_+ + \delta I - |B|\,\| \leqslant \delta$, so that $\|A - S_\delta\| \leqslant 2\delta$.

(2) If $A \in L_{\mathrm{s}}$ then there exists an operator $B = V|B| \in L_0^{-1} \cap L_{\mathrm{s}}$ such that $\|A - B\| \leqslant \delta$. As in (1), we can take $S_\delta := V((|B| - \delta I)_+ + \delta I) \in L_{\mathrm{s}}$.

(3) Let $S := U|S|$ be the polar representation of S. By Lemma 1.7, $U \in L_0^{-1} \cap L_{\mathrm{u}}$. Therefore $A := W_\varepsilon\, U_\varepsilon\, |S|$, where W_ε and U_ε are unitary operators satisfying the conditions of Lemma 1.8.

Let $Z_1(t) := \exp(it \operatorname{Arg} W_\varepsilon)$, $Z_2(t) := \exp(it \operatorname{Arg} U_\varepsilon)$ and $Z_3(t) := t|A_\delta| + (1-t)I$, where $t \in [0,1]$. Each $Z_j(t)$ is a path in L_0^{-1}, and so is $Z(t) := Z_1(t)Z_2(t)Z_3(t)$. Obviously, $Z(0) = I$, $Z(1) = S$ and

$$\|Z(t)^{-1}\| \;=\; \|(t|S| + (1-t)I)^{-1}\| \;=\; \big(t\|S^{-1}\|^{-1} + (1-t)\big)^{-1} \;\leqslant\; \max\big\{1\,,\,\|S^{-1}\|\big\}\,.$$

One can easily see that $\|Z_3(t) - Z_3(r)\| \leqslant |t-r|\,\max\{1\,,\,\|S\|\}$, $\|Z_3(t)\| \leqslant \max\{1\,,\,\|S\|\}$ and $\|Z_1(t)\| = \|Z_2(t)\| = 1$. Since $|e^{it\theta} - e^{ir\theta}| \leqslant \pi\,|t-r|$ for all $r,t \in \mathbb{R}$ and $\theta \in (-\pi,\pi)$, we also have $\|Z_j(t) - Z_j(r)\| \leqslant \pi\,|t-r|$ for $j = 1,2$. These inequalities imply (1.1). □

2. Main results

2.1. **Resolution of the identity.** The following theorem will be proved in Appendix A. Roughly speaking, it says that a normal operator $A \in L_{\mathrm{n}}$ satisfying the condition **(C)** has a resolution of the identity in L associated with any finite open cover of $\sigma(A)$.

Theorem 2.1. *Assume that L has real rank zero. Let $A \in L_{\mathrm{n}}$, and let $\{\Omega_j\}_{j=1}^m$ be a finite open cover of $\sigma(A)$. If A satisfies the condition* **(C)** *then there exists a family of mutually orthogonal projections $P_j \in L$ such that*

$$\sum_{j=1}^m P_j = I \quad \text{and} \quad P_j H \subset \Pi_{\Omega_j} H \text{ for all } j = 1,\dots,m\,, \tag{2.1}$$

where Π_{Ω_j} are the spectral projections of A corresponding to the sets Ω_j.

Remark 2.2. The operators P_j can be thought of as approximate spectral projections of A. If L is a von Neumann algebra then the spectral projections of A belong to L and one can simply take $P_j = \Pi_{\Omega'_j}$, where $\{\Omega'_j\}_{j=1}^m$ is an arbitrary collection of mutually disjoint subsets $\Omega'_j \subset \Omega_j$ covering $\sigma(A)$. However, even in this situation Theorem 2.1 may be useful, since the projections P_j constructed in the proof continuously depend on A in the norm topology.

The following simple lemma shows how Theorem 2.1 can be applied for approximation purposes.

Lemma 2.3. *Let $A \in L_{\mathrm{n}}$, and let $\{\Omega_j\}_{j=1}^m$ be a finite open cover of $\sigma(A)$ whose multiplicity does not exceed k. If there exist mutually orthogonal projections P_j satisfying* (2.1) *then*

$$\|A - \sum_{j=1}^m z_j\, P_j\| \;\leqslant\; \sqrt{k}\,\max_j(\operatorname{diam}\,\Omega_j) \tag{2.2}$$

for any collection of points $z_j \in \Omega_j$.

Proof. If $z_j \in \Omega_j$ then

$$
\begin{aligned}
\|Au - \sum_{j=1}^{m} z_j\, P_j u\|^2 \;&=\; \sum_{j=1}^{m} \|P_j A u - z_j\, P_j u\|^2 \;\leqslant\; \sum_{j=1}^{m} \|\Pi_{\Omega_j}(Au - z_j u)\|^2 \\
&=\; \sum_{j=1}^{m} \|(A - z_j I)\,\Pi_{\Omega_j} u\|^2 \;\leqslant\; \sum_{j=1}^{m} \|\Pi_{\Omega_j} u\|^2\,(\mathrm{diam}\;\Omega_j)^2 \;\leqslant\; k\,\|u\|^2\,\max_j(\mathrm{diam}\;\Omega_j)^2
\end{aligned}
$$

for all $u \in H$. Taking the supremum over u, we obtain (2.2). □

Theorem 2.2 and Lemma 2.3 imply the following corollaries.

Corollary 2.4. *The following statements are equivalent.*

1. *A C^*-algebra L has real rank zero.*
2. *Every self-adjoint operator $A \in L_{\mathrm{s}}$ has approximate spectral projections in the sense of Theorem* 2.1*, associated with any finite open cover of its spectrum.*
3. $L_{\mathrm{s}} = \overline{L_{\mathrm{f}} \cap L_{\mathrm{s}}}$.

Corollary 2.5. *Assume that L has real rank zero. Then for every normal operator $A \in L_{\mathrm{n}}$ the following statements are equivalent.*

1. *The operator A satisfies the condition* **(C)***.*
2. *The operator A has approximate spectral projections in the sense of Theorem* 2.1*, associated with any finite open cover of its spectrum.*
3. $A \in \overline{L_{\mathrm{f}} \cap L_{\mathrm{n}}}$.

Proof. The corollaries are proved in the same manner.

By Remark 1.6, every self-adjoint operator $A \in L_{\mathrm{s}}$ satisfies the condition **(C)**. Therefore the implications $(1) \Rightarrow (2)$ follow from Theorem 2.1.

Any subset of $\mathbb{C}$ admits a cover $\{\Omega_j\}_{j=1}^m$ of multiplicity four by open squares Ω_j of arbitrarily small size. If $A \in L_{\mathrm{n}}$ has approximate spectral projections P_j associated with all such covers of its spectrum then, in view of (2.2), the operator A can be approximated by operators of the form $\sum_{j=1}^m z_j\, P_j \in L_{\mathrm{f}} \cap L_{\mathrm{n}}$. Moreover, if $A \in L_{\mathrm{s}}$ then we can take $z_j \in \mathbb{R}$, so that $\sum_{j=1}^m z_j\, P_j \in L_{\mathrm{f}} \cap L_{\mathrm{s}}$. Thus $(2) \Rightarrow (3)$.

Finally, in view of Remark 1.2, every operator $T \in L_{\mathrm{f}} \cap L_{\mathrm{n}}$ can be written in the form $\sum_{j=1}^m z_j\, \Pi_j$, where $z_j \in \mathbb{R}$ whenever $T \in L_{\mathrm{s}}$ and Π_j are mutually orthogonal projections lying in L. If A is approximated by a sequence of such operators then it can also be approximated by a sequence of operators $\sum_{j=1}^m \tilde z_j\, \Pi_j$, where Π_j are the same projections, $\tilde z_j \neq 0$ and $\mathrm{Im}\,\tilde z_j = \mathrm{Im}\, z_j$. This shows that $(3) \Rightarrow (1)$. □

Remark 2.6. The implications $(1) \Leftrightarrow (3)$ in the above corollaries are known results (see Subsection B.1 and [FR2, Theorem 3.2]). In [FR2], the authors explained that the part $(1) \Rightarrow (3)$ of Corollary 2.5 would follow from the existence of projections 'that approximately commute with A and approximately divide $\sigma(A)$ into disjoint components'. The implications $(1) \Rightarrow (2) \Rightarrow (3)$ in Corollary 2.5 give a precise meaning to their statement.

2.2. **The main theorem.** In this subsection

- $\mathrm{B}(r) := \{T \in L : \|T\| \leqslant r\}$ is the closed ball about the origin in L of radius r;
- M_T denotes the convex hull of the set $\bigcup_{S_1,S_2\in\mathrm{B}(1)} S_1TS_2$, where $T \in L$;
- J_T is the two-sided ideal in L generated by the operator $T \in L$.

Remark 2.7. The ideal J_T consists of finite linear combinations of operators of the form S_1TS_2 where $S_1, S_2 \in L$. Therefore $J_T = \bigcup_{t\geqslant 0} tM_T$ and $M_T \subset J_T \cap \mathrm{B}(\|T\|)$.

Remark 2.8. The unit ball $\mathrm{B}(1)$ coincides with the closed convex hull of L_{u} (see, for example, [RD]). This implies that M_T is a subset of the closed convex hull of the set $\bigcup_{U,V\in L_{\mathrm{u}}} UTV$. Moreover, if $\overline{L^{-1}} = L$ then $\mathrm{B}(1)$ coincides with the convex hull of L_{u} (see [R]) and, consequently, every element of M_T is a finite convex combination of operators of the form UTV with $U, V \in L_{\mathrm{u}}$.

We shall say that a continuous real valued function f satisfies the condition $(\mathrm{C}_{\varepsilon,r})$ for some $\varepsilon, r > 0$ if f is defined on the interval $[-r-\varepsilon, r+\varepsilon]$ and

$(\mathrm{C}_{\varepsilon,r})$ there exists $\varepsilon' \in (0,\varepsilon)$ such that the set $\{x \in \mathbb{R} : f(x) = y\}$ is an ε'-net in $\{x \in \mathbb{R} : x^2 + y^2 \leqslant (r+\varepsilon)^2\}$ for each $y \in [-r-\varepsilon, r+\varepsilon]$.

The condition $(\mathrm{C}_{\varepsilon,r})$ is fulfilled whenever the function f sufficiently rapidly oscillates between $-r-\varepsilon$ and $r+\varepsilon$. In particular, it holds for $f(x) = (r+\varepsilon)\cos(\pi x/\varepsilon)$.

Lemma 2.9. *Let L have real rank zero, and let $X, Y \in L_{\mathrm{s}}$ be self-adjoint operators such that the operator $A := X + iY$ satisfies the condition* **(C)**. *Then, for every function f satisfying the condition $(\mathrm{C}_{\varepsilon,r})$ with $r = \|A\|$, the operator Y belongs to the closure of the set $f\,(X + \mathrm{B}(\varepsilon) \cap L_{\mathrm{s}}) + J_{[X,Y]} \cap L_{\mathrm{s}}$.*

Proof. Assume first that $J_{[X,Y]} = \{0\}$, so that A is normal. By Corollary 2.5, for each $\delta \in (0,\varepsilon]$ there exists an operator $A_\delta \in L_{\mathrm{n}} \cap L_{\mathrm{f}}$ with finite spectrum $\sigma(A_\delta) = \{z_1, \dots, z_m\}$ such that $\|A - A_\delta\| \leqslant \delta$ and, consequently, $|z_j| \leqslant r + \delta$ for all j. In view of $(\mathrm{C}_{\varepsilon,r})$, one can find real numbers $\varepsilon_k \in [-\varepsilon', \varepsilon']$ such that $z_k + \varepsilon_k$ lie on the graph of f for all $k = 1, \dots, m$. Let A'_δ be the operator with eigenvalues $z_k + \varepsilon_k$ and the same spectral projections as A_δ. Then $\|Y - \operatorname{Im} A'_\delta\| \leqslant \delta$, $\|X - \operatorname{Re} A'_\delta\| \leqslant \delta + \varepsilon'$ and $\operatorname{Im} A'_\delta = f(\operatorname{Re} A'_\delta)$. This implies that

$$(Y + \mathrm{B}(\delta)) \bigcap f\,(X + \mathrm{B}(\varepsilon) \cap L_{\mathrm{s}}) \neq \varnothing, \qquad \forall \delta \in (0, \varepsilon - \varepsilon'].$$

Letting $\delta \to 0$, we see that $Y \in \overline{f\,(X + \mathrm{B}(\varepsilon) \cap L_{\mathrm{s}})}$.

Assume now that $J_{[X,Y]} \neq \{0\}$ and denote $L' := \overline{J_{[X,Y]}}$. Let us consider the quotient C^*-algebra L/L' and the corresponding quotient map $\pi : L \mapsto L/L'$. Since the map π is continuous and $\pi S = \pi(S + S^*)/2$ for all self-adjoint elements $\pi S \in L/L'$, the quotient algebra also has real rank zero and $\pi S \in \overline{(L/L')_0^{-1}}$ whenever $S \in \overline{L_0^{-1}}$. The latter implies that the normal element πA of the quotient algebra L/L' also satisfies the condition **(C)**.

Applying the previous result with ε replaced by $\varepsilon_0 \in (\varepsilon', \varepsilon)$ to $\pi A = \pi X + i\pi Y$, we can find a sequence of operators $X_n \in L_{\mathrm{s}}$ such that $f(\pi X_n) \to \pi Y$ as $n \to \infty$ and $\|\pi X - \pi X_n\| \leqslant \varepsilon_0$ for all n. Since $\|T\| \geqslant \|\operatorname{Re} T\|$ for all $T \in L$, we have

$$\|\pi T\| := \inf_{R\in L'} \|T - R\| = \inf_{R\in L'\cap L_{\mathrm{s}}} \|T - R\|, \qquad \forall T \in L_{\mathrm{s}}. \tag{2.3}$$

Therefore, there exist operators $R_n \in L' \cap L_{\rm s}$ such that $X_n + R_n \in X + {\rm B}(\varepsilon) \cap L_{\rm s}$. Since f can be uniformly approximated by polynomials on any compact subset of $\mathbb{R}$ and $Q(\pi X_n) = \pi Q(X_n)$ for any polynomial Q, we have $f(\pi X_n) = f(\pi(X_n + R_n)) = \pi f(X_n + R_n)$ and, consequently, $\|\pi(f(X_n+R_n)-Y)\| \to 0$. In view of (2.3), there exist operators $\tilde R_n \in L' \cap L_{\rm s}$ such that $f(X_n+R_n)+\tilde R_n \to Y$ as $n \to \infty$. This implies that Y belongs to the closure of the set $f\,(X + {\rm B}(\varepsilon) \cap L_{\rm s}) + L' \cap L_{\rm s}$ which coincides with $\overline{f\,(X + {\rm B}(\varepsilon) \cap L_{\rm s}) + J_{[X,Y]} \cap L_{\rm s}}$. $\square$

Corollary 2.10. *Assume that L has real rank zero. If $A \in L$ satisfies* **(C)** *then*

$$A \;\in\; \overline{{\rm B}(\|A\|) \cap L_{\rm n} + J_{[A^*,A]} \cap L_{\rm s}}\,. \tag{2.4}$$

Proof. Let $r := \|A\|$. Given $\varepsilon > 0$, let us choose a function f_ε satisfying the condition $({\rm C}_{\varepsilon,r})$ whose graph lies in the disc $\{x^2 + y^2 \leqslant (r+\varepsilon)^2\}$. Applying Lemma 2.9, we can find an operator $X_\varepsilon \in {\rm Re}\, A + {\rm B}(\varepsilon) \cap L_{\rm s}$ such that ${\rm Im}\, A \in f_\varepsilon(X_\varepsilon) + J_{[A^*,A]} \cap L_{\rm s} + {\rm B}(\varepsilon) \cap L_{\rm s}$. The operator $\tilde A_\varepsilon := X_\varepsilon + i f_\varepsilon(X_\varepsilon)$ is normal, and $A - \tilde A_\varepsilon \in J_{[A^*,A]} \cap L_{\rm s} + {\rm B}(2\varepsilon) \cap L_{\rm s}$. Therefore, there exist operators $R_\varepsilon \in J_{[A^*,A]} \cap L_{\rm s}$ such that $\tilde A_\varepsilon + R_\varepsilon \to A$ as $\varepsilon \to 0$. Since $x^2 + (f(x))^2 \leqslant (r+\varepsilon)^2$, we have

$$\|\tilde A_\varepsilon\|^2 \;=\; \|X_\varepsilon\|^2 + \|f_\varepsilon(X_\varepsilon)\|^2 \;\leqslant\; (r+\varepsilon)^2\,.$$

If $A_\varepsilon := r\,(r+\varepsilon)^{-1}\tilde A_\varepsilon$ then, by the above, $A_\varepsilon \in {\rm B}(r) \cap L_{\rm n}$ and $A_\varepsilon + R_\varepsilon \to A$ as $\varepsilon \to 0$. $\square$

Remark 2.11. By Corollary 2.10, $\overline{L_{\rm n} + L'} = \overline{L_{\rm n} + L' \cap L_{\rm s}}$ for any two-sided ideal $L' \subset L$ in a C^*-algebra L of real rank zero. Indeed, if $A \in L_{\rm n} + L'$ then $J_{[A^*,A]} \subset L'$ and, in view of (2.4), A can be approximated by operators from $L_{\rm n} + L' \cap L_{\rm s}$.

The following refinement of Corollary 2.10 is the main result of the paper.

Theorem 2.12. *There is a nonincreasing function $h : (0,\infty) \mapsto [0,\infty)$ such that $h(\varepsilon) = 0$ for all $\varepsilon \geqslant 1$ and*

$$A \;\in\; {\rm B}(\|A\|) \cap L_{\rm n} \;+\; h(\varepsilon)\, M_{[A^*,A]} \cap L_{\rm s} \;+\; {\rm B}(\varepsilon) \tag{2.5}$$

for all $\varepsilon \in (0,\infty)$, all C^*-algebras L of real rank zero and all operators $A \in {\rm B}(1)$ satisfying the condition **(C)**.

Remark 2.13. In other words, the inclusion (2.5) means that for each $\varepsilon > 0$ there exist a normal operator $T(\varepsilon) \in L_{\rm n}$ and a finite linear combination

$$S(\varepsilon) \;=\; \sum_j c(j,\varepsilon)\, S_1(j,\varepsilon)[A^*,A]S_2(j,\varepsilon) \tag{2.6}$$

with $S_k(j,\varepsilon) \in L$ and $c(j,\varepsilon) \in (0,1]$ such that $\|T(\varepsilon)\| \leqslant \|A\|$, $S(\varepsilon) \in L_{\rm s}$, $\|S_k(j,\varepsilon)\| \leqslant 1$, $\sum_j c(j,\varepsilon) = 1$ and

$$\|\, A - T(\varepsilon) - h(\varepsilon)\, S(\varepsilon)\, \| \;\leqslant\; \varepsilon\,.$$

Note that (2.6) can be written as a linear combination of self-adjoint operators,

$$S(\varepsilon) \;=\; \sum_j c(j,\varepsilon)\, \big(S_+^*(j,\varepsilon)[A^*,A]S_+(j,\varepsilon) - S_-^*(j,\varepsilon)[A^*,A]S_-(j,\varepsilon)\big) \tag{2.7}$$

where $\|S_\pm(j,\varepsilon)\| \leqslant 1$. Indeed, if $S_\pm(j,\varepsilon) := \frac{1}{2}\,(S_1^*(j,\varepsilon) \pm S_2(j,\varepsilon))$ then the real part of each term in the right hand side of (2.6) coincides with corresponding term in (2.7).

Proof. Let us consider a family of C^*-algebras L_ξ of real rank zero parameterised by $\xi \in \Xi$, where Ξ is an arbitrary index set, and let $\mathcal{L}$ be their direct product. By definition, the C^*-algebra $\mathcal{L}$ consists of families $\mathcal{S} = \{S_\xi\}$ with $S_\xi \in L_\xi$ such that $\|\mathcal{S}\|_{\mathcal{L}} := \sup_{\xi\in\Xi} \|S_\xi\| < \infty$, $\mathcal{S}^* := \{S_\xi^*\}$ and $\mathcal{S}\tilde{\mathcal{S}} = \{S_\xi \tilde{S}_\xi\}$. Let $\mathrm{B}_{\mathcal{L}}(r)$ and $\mathrm{B}_\xi(r)$ be the balls of radius r about the origin in $\mathcal{L}$ and L_ξ respectively.

In view of Lemma 1.9(2), $\mathcal{L}$ has real rank zero. Lemma 1.9(3) implies that $\{S_\xi\} \in \mathcal{L}_0^{-1}$ whenever $\{S_\xi\} \in \mathcal{L}$, $S_\xi \in (L_\xi)_0^{-1}$ for each $\xi \in \Xi$ and $\sup_{\xi\in\Xi} \|S_\xi^{-1}\| < \infty$. From here and Lemma 1.9(1) it follows that $\mathcal{A} = \{A_\xi\} \in \mathcal{L}$ satisfies the condition **(C)** whenever all the operators A_ξ satisfy **(C)**.

Let us fix $\varepsilon \in (0,1)$ and consider an arbitrary family $\mathcal{A} = \{A_\xi\} \in \mathcal{L}$ of operators $A_\xi \in L_\xi$ satisfying **(C)**. Applying Corollary 2.10 to $\mathcal{A}$, we see that there exist families of operators $\mathcal{T}_\varepsilon = \{T_{\xi,\varepsilon}\} \in \mathrm{B}_{\mathcal{L}}(\|\mathcal{A}\|_{\mathcal{L}}) \cap \mathcal{L}_{\mathrm{n}}$ and $\mathcal{R}_\varepsilon = \{R_{\xi,\varepsilon}\} \in J_{[\mathcal{A}^*,\mathcal{A}]} \cap \mathcal{L}_{\mathrm{s}}$ such that $\|\mathcal{A} - \mathcal{T}_\varepsilon - \mathcal{R}_\varepsilon\|_{\mathcal{L}} \leqslant \varepsilon$. The estimate for the norm holds if and only if $A_\xi - T_{\xi,\varepsilon} - R_{\xi,\varepsilon} \in \mathrm{B}_\xi(\varepsilon)$ for all $\xi \in \Xi$. The inclusion $\mathcal{T}_\varepsilon \in \mathrm{B}_{\mathcal{L}}(\|\mathcal{A}\|_{\mathcal{L}}) \cap \mathcal{L}_{\mathrm{n}}$ means that $T_{\xi,\varepsilon} \in \mathrm{B}_\xi(\|\mathcal{A}\|_{\mathcal{L}}) \cap (L_\xi)_{\mathrm{n}}$ for all $\xi \in \Xi$. Finally, by Remark 2.7, $J_{[\mathcal{A}^*,\mathcal{A}]} = \bigcup_{t\geqslant 0} tM_{[\mathcal{A}^*,\mathcal{A}]}$. This identity and the inclusion $\mathcal{R}_\varepsilon \in J_{[\mathcal{A}^*,\mathcal{A}]} \cap \mathcal{L}_{\mathrm{s}}$ imply that $R_{\xi,\varepsilon} \in tM_{[A_\xi^*,A_\xi]} \cap (L_\xi)_{\mathrm{s}}$ for all $\xi \in \Xi$ and some t independent of ξ. Thus we obtain

$$A_\xi \ \in \ \mathrm{B}_\xi(\|\mathcal{A}\|_{\mathcal{L}}) \cap (L_\xi)_{\mathrm{n}} \ + \ tM_{[A_\xi^*,A_\xi]} \cap (L_\xi)_{\mathrm{s}} \ + \ \mathrm{B}_\xi(\varepsilon)\,, \qquad \forall \xi \in \Xi\,, \tag{2.8}$$

where t is a nonnegative number which does not depend on ξ.

If (2.5) were not true for any $h(\varepsilon) \in [0,\infty)$ then there would exist families of C^*-algebras L_ξ and operators $A_\xi \in L_\xi$ satisfying the condition **(C)**, for which (2.8) would not hold with any t independent of ξ. However, by the above, it is not possible. Thus we have (2.5) with some function h for all $\varepsilon \in (0,1)$. Since $A \in \mathrm{B}(1)$, we can extend $h(\varepsilon)$ by zero for $\varepsilon \geqslant 1$. It remains to notice that h can be chosen nonincreasing because the same inclusion holds for $\tilde{h}(\varepsilon) := \sup_{t\geqslant\varepsilon} h(t)$. □

If $t \in [0,\infty)$, let

$$F(t) \ := \ \inf_{\varepsilon>0} \left(h(\varepsilon)\, t + \varepsilon\right)\,, \tag{2.9}$$

where h is the function introduced in Theorem 2.12. The function $F : [0,\infty) \mapsto [0,1]$ is nondecreasing, $F(0) = 0$ and $F(t) > 0$ whenever $t > 0$. Since the subgraph of F coincides with an intersection of half-planes, F is concave and, consequently, continuous.

Corollary 2.14. *Let L be a C^*-algebra of real rank zero, and let $\|\cdot\|_\star$ be a continuous seminorm on L such that*

$$\|USV\|_\star \ \leqslant \ \|S\|_\star \quad \textit{and} \quad \|S\|_\star \leqslant C_\star \|S\| \quad \textit{for all } S \in L \textit{ and all } U,V \in L_{\mathrm{u}}\,, \tag{2.10}$$

where $C_\star$ is a positive constant. Then

$$\inf_{T\in L_{\mathrm{n}}\,:\,\|T\|\leqslant\|A\|} \|A - T\|_\star \ \leqslant \ C_\star\, F(C_\star^{-1} \|[A^*,A]\|_\star) \tag{2.11}$$

for all operators $A \in \mathrm{B}(1)$ satisfying the condition **(C)**.

Proof. In view of Remark 2.8, from the inequalities (2.10) it follows that

$$\|S_1 S S_2\|_\star \;\leqslant\; \|S_1\|\,\|S\|_\star\,\|S_2\| \quad \text{for all } S, S_1, S_2 \in L \tag{2.12}$$

and, consequently, $\|S\|_\star \leqslant \|[A^*,A]\|_\star$ for all $S \in M_{[A^*,A]}$. Since $\|R\|_\star \leqslant C_\star\,\|R\| \leqslant \varepsilon\, C_\star$ for all $R \in \mathrm{B}(\varepsilon)$, the inclusion (2.5) implies that

$$\inf_{T \in L_{\mathrm{n}}\,:\,\|T\|\leqslant\|A\|} \|A-T\|_\star \;\leqslant\; h(\varepsilon)\,\|[A^*,A]\|_\star + \varepsilon\, C_\star\,, \qquad \forall \varepsilon > 0\,.$$

Taking the infimum over $\varepsilon > 0$, we obtain (2.11). □

Remark 2.15. It is clear from the prooof that (2.11) can be extended to general functions $\|\cdot\|_\star : L \mapsto \mathbb{R}_+$ satisfying (2.10) and suitable quasiconvexity conditions.

Example 2.16. Let J be a two-sided ideal in L. Then the seminorm $\|A\|_\star := \operatorname{dist}(A, J)$ satisfies the conditions (2.10) with $C_\star = 1$. Corollary 2.14 implies that

$$\operatorname{dist}\bigl(A, J + \mathrm{B}(\|A\|) \cap L_{\mathrm{n}}\bigr) \;\leqslant\; F\bigl(\operatorname{dist}([A^*,A], J)\bigr) \tag{2.13}$$

for all $A \in \mathrm{B}(1)$ satisfying the condition **(C)**.

3. Applications

Throughout this section

- $\mathcal{C}(H)$ is the C^*-algebra of compact operators in H;
- $\mathcal{S}_p$ are the Schatten classes of compact operators and
- $\|\cdot\|_p$ are the corresponding norms (we shall always be assuming that $p \geqslant 1$);
- $\|A\|_{\mathrm{ess}} := \inf_{K\in\mathcal{C}(H)} \|A-K\|$ is the distance from A to $\mathcal{C}(H)$;
- F is the function defined by (2.9).

3.1. **Matrices.** Let L be the linear space of all complex $m\times m$ matrices. Then the Schatten norms $\|\cdot\|_p$ on L satisfy (2.10) with $C_\star = m^{1/p}$. Corollary 2.14 implies that

$$\inf_{T\in L_{\mathrm{n}}} \|A-T\| \;\leqslant\; \inf_{T \in L_{\mathrm{n}}\,:\,\|T\|\leqslant\|A\|} \|A-T\| \;\leqslant\; F\bigl(\|[A^*,A]\|\bigr) \tag{3.1}$$

and

$$\inf_{T\in L_{\mathrm{n}}} \|A-T\|_p \;\leqslant\; \inf_{T \in L_{\mathrm{n}}\,:\,\|T\|\leqslant\|A\|} \|A-T\|_p \;\leqslant\; m^{1/p}\, F\bigl(m^{-1/p}\|[A^*,A]\|_p\bigr) \tag{3.2}$$

for all $p\in[1,\infty)$, all $m = 1, 2, \dots$ and all $A\in L$ such that $\|A\|\leqslant 1$.

Note that the $\mathcal{S}_2$-distance from a given $m\times m$-matrix A to the set of normal matrices admits the following simple description.

Lemma 3.1. *Let A be an $m\times m$-matrix, and let $\Sigma_m(A)$ be the set of all complex vectors $\mathbf{z}\in\mathbb{C}^m$ of the form*

$$\mathbf{z} \;=\; \{(Au_1,u_1), (Au_2,u_2)\dots,(Au_m,u_m)\}$$

where $\{u_1,u_2,\dots,u_m\}$ is an orthonormal basis. Then

$$\inf_{T\in L_{\mathrm{n}}} \|A-T\|_2^2 \;=\; \inf_{T \in L_{\mathrm{n}}\,:\,\|T\|\leqslant\|A\|} \|A-T\|_2^2 \;=\; \|A\|_2^2 \;-\; \sup_{\mathbf{z}\in\Sigma_m(A)} |\mathbf{z}|^2\,. \tag{3.3}$$

Proof. If T is an arbitrary normal matrix and $\{u_1, u_2, \ldots, u_m\}$ is the basis formed by its eigenvectors then

$$\|A-T\|_2^2 \;\geqslant\; \sum_{j\neq k} |(Au_j,u_k)|^2 \;=\; \|A\|_2^2 - \sum_{j=1}^m |(Au_j,u_j)|^2 \;\geqslant\; \|A\|_2^2 \;-\; \sup_{\mathbf{z}\in\Sigma_m(A)} |\mathbf{z}|^2\,. \tag{3.4}$$

Therefore $\inf\limits_{T\in L_{\mathrm{n}}} \|A-T\|_2^2 \;\geqslant\; \|A\|_2^2 \;-\; \sup\limits_{\mathbf{z}\in\Sigma_m(A)} |\mathbf{z}|^2\,.$

On the other hand, since the set $\Sigma_m(A)$ is compact, we have $\sup_{\mathbf{z}\in\Sigma_m(A)}|\mathbf{z}|^2 = |\mathbf{z}_0|^2$ for some $\mathbf{z}_0\in\Sigma_m(A)$. Let us write down the matrix A in a corresponding orthonormal basis $\{v_1,\ldots,v_m\}$ and denote by T_0 the normal matrix obtained by removing the off-diagonal elements. Then $\|T_0\|\leqslant\|A\|$ and $\|A-T_0\|_2^2 = \sum_{j\neq k}|(Av_j,v_k)|^2 = \|A\|_2^2 - |\mathbf{z}_0|^2$. □

The following example shows that for $p=2$ the estimate (3.2) is order sharp as $m\to\infty$.

Example 3.2. Let m be even, and let $\{e_j\}_{j=1}^m$ be the standard Euclidean basis in $\mathbb{C}^m$. Consider the $m\times m$-matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0\\ 0 & 0 & 0 & 0 & \ldots & 0 & 0\\ 0 & 0 & 0 & 1 & \ldots & 0 & 0\\ 0 & 0 & 0 & 0 & \ldots & 0 & 0\\ . & . & . & . & \ldots & . & .\\ 0 & 0 & 0 & 0 & \ldots & 0 & 1\\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

defined by the identities $Ae_{2i}=e_{2i-1}$ and $Ae_{2i-1}=0$, where $i=1,\ldots,m/2$. By direct calculation, $\|A\|=1$ and $[A,A^*]=\operatorname{diag}(1,-1,\ldots,1,-1,)$, so that $\|[A,A^*]\|_2 = m^{1/2}$.

For this matrix A,

$$\inf_{T\in L_{\mathrm{n}}} \|A-T\|_2 \;=\; (m/4)^{1/2} \;=\; m^{1/2}\left(2^{-1}m^{-1/2}\|[A,A^*]\|_2\right)\,.$$

Indeed, let $\{u_j\}_{j=1}^m$ be an orthonormal basis in $\mathbb{C}^m$. Then $u_j=\sum_{i=1}^{m/2}(\alpha_{i,j}\,e_{2i-1}+\beta_{i,j}\,e_{2i})$, where $\alpha_{i,j}$ and $\beta_{i,j}$ are complex numbers such that $\sum_{i=1}^{m/2}(|\alpha_{i,j}|^2+|\beta_{i,j}|^2)=1$. Clearly,

$$(Au_j,u_j) \;=\; \sum_{i=1}^{m/2}(\beta_{i,j}\,e_{2i-1},u_j) \;=\; \sum_{i=1}^{m/2}\beta_{i,j}\,\overline{\alpha_{i,j}}\,.$$

Therefore, $2\,|(Au_j,u_j)|\leqslant\sum_{i=1}^{m/2}(|\alpha_{i,j}|^2+|\beta_{i,j}|^2)=1$ and $\sum_{j=1}^m|(Au_j,u_j)|^2\leqslant m/4$. Thus we have $|\mathbf{z}|^2\leqslant m/4$ for all $\mathbf{z}\in\Sigma_m(A)$. Since $\|A\|_2^2=m/2$, Lemma 3.1 implies that $\|A-T\|_2^2\geqslant m/4$ for all normal matrices T. On the other hand, if $T_0=\operatorname{Re}A$ then $\|A-T_0\|_2^2=\|\operatorname{Im}A\|_2^2=m/4$.

Remark 3.3. If T is a normal matrix and A is an arbitrary matrix of the same size then

$$[A^*,A] \;=\; [A^*,T]+[A^*,A-T] \;=\; [(A-T)^*,T]+[A^*,(A-T)]\,.$$

Estimating the Schatten norms of the right and left hand sides, we obtain

$$\|[A^*,A]\|_p \;\leqslant\; 2\,(\|A\|+\|T\|)\,\|A-T\|_p\,.$$

This implies that

$$\inf_{T\in L_{\mathrm{n}}\,:\,\|T\|\leqslant\|A\|}\|A-T\|_p \;\geqslant\; \frac{\|[A^*,A]\|_p}{4\|A\|} \tag{3.5}$$

for all finite matrices A and all $p\in[1,\infty]$.

Substituting the matrix A from Example 3.2 into (3.5), we see that the second estimate (3.2) is order sharp as $m\to\infty$ for all $p\in[1,\infty)$.

3.2. **Bounded and compact operators.** If L is the C^*-algebra obtained from $\mathcal{C}(H)$ by adjoining the unity then, by Remark 1.6, $L^{-1}=L_0^{-1}$ and all $A\in L$ satisfy the condition **(C)**. Thus our main results hold for all compact operators A.

If $L=\mathcal{B}(H)$ then $L^{-1}=L_0^{-1}$ because every unitary operator can be joined with I by the path $\exp(it\operatorname{Arg}U)$ (see Lemma 1.7). However, in the infinite dimensional case $\overline{L^{-1}}\neq\mathcal{B}(H)$. The following result was obtained in [FK] (it also follows from [CL, Theorem 4.1] or [Bo1, Theorem 3]).

Lemma 3.4. *Let H be separable, and let $L=\mathcal{B}(H)$. Then an operator A satisfies the condition* **(C)** *if and only if for each $\lambda\in\mathbb{C}$ either the range $(A-\lambda I)H$ is not closed or* $\dim\ker(A-\lambda I)=\dim\ker(A^*-\bar\lambda I)$.

In other words, Lemma 3.4 states that in the separable case **(C)** is equivalent to the condition on the index function in the BDF theorem. In particular, this implies that normal operators and their compact perturbations satisfy the condition **(C)**.

Remark 3.5. An explicit description of the closure of the set of invertible operators in a nonseparable Hilbert space was obtained in [Bo2].

3.3. **The BDF theorem.** In this subsection we are always assuming that H is separable and $L=\mathcal{B}(H)$.

Recall that an operator $A\in\mathcal{B}(H)$ is called *quasidiagonal* if it can be represented as the sum of a block diagonal and a compact operator, that is, if there exist mutually orthogonal finite dimensional subspaces H_k and operators $S_k:H_k\mapsto H_k$ such that $H=\bigoplus_{k=1}^\infty H_k$ and $A=\operatorname{diag}\{S_1,S_2,\dots\}+K$, where $K\in\mathcal{C}(H)$.

We shall need the following well known result.

Lemma 3.6. *The set of compact perturbations of normal operators on a separable Hilbert space is norm closed and coincides with the set of quasidiagonal operators $S\in\mathcal{B}(H)$ such that $[S^*,S]\in\mathcal{C}(H)$.*

Lemma 3.6 follows from the BDF theorem but it also admits a simple independent proof based on Theorem 0.2 (see [FR2, Proposition 2.8]). Obviously, the BDF theorem is an immediate consequence of Corollary 2.10 and Lemma 3.6. One obtains a slightly better result by applying the following lemma, which shows that the BDF theorem holds with a normal operator T such that $\|T\|\leqslant\|A\|$.

Lemma 3.7. *Let H be separable, and let $L=\mathcal{B}(H)$. Then, for each fixed $r>0$, the set $B(r)\cap L_n+\mathcal{C}(H)$ is closed and coincides with the set of quasidiagonal operators* $\operatorname{diag}\{S_1,S_2,\dots\}+K$ *such that $K\in\mathcal{C}(H)$, S_k are normal and $\|S_k\|\leqslant r$ for all k.*

Proof. Obviously, if $A = \operatorname{diag}\{S_1, S_2, \ldots\} + K$ then $A \in B(r) \cap L_n + \mathcal{C}(H)$ whenever S_k and K satisfy the conditions of the lemma.

Assume now that $A \in \overline{B(r) \cap L_n + \mathcal{C}(H)}$. Then $\|A\|_{\mathrm{ess}} \leqslant r$, $[A^*, A] \in \mathcal{C}(H)$ and, by Lemma 3.6, $A = \operatorname{diag}\{S'_1, S'_2, \ldots\} + K'$, where $K' \in \mathcal{C}(H)$ and S'_k are operators acting in some mutually orthogonal finite dimensional subspaces H_k such that $H = \bigoplus_k H_k$. Since the self-commutator $[A^*, A]$ is compact, we have $[(S'_k)^*, S'_k] \to 0$ as $k \to \infty$. By (3.1), there are normal operators $S''_k : H_k \mapsto H_k$ such that $\|S'_k - S''_k\| \to 0$ as $k \to \infty$. The operator $\operatorname{diag}\{S'_1 - S''_1, S'_2 - S''_2, \ldots\}$ is compact, so that $A = \operatorname{diag}\{S''_1, S''_2, \ldots\} + K''$ where $K'' \in \mathcal{C}(H)$.

Since $\|A\|_{\mathrm{ess}} \leqslant r$, we have $\limsup_{k\to\infty} \|S''_k\| \leqslant r$. Define

$$S_k \; := \; \begin{cases} S''_k\,, & \text{if } \|S''_k\| \leqslant r\,,\\ r\,\|S''_k\|^{-1}\, S''_k\,, & \text{if } \|S''_k\| > r\,. \end{cases}$$

Clearly, S_k are normal and $\|S_k\| \leqslant r$. The estimate for the upper limit implies that

$$\limsup_{k\to\infty} \|S''_k - S_k\| \;=\; \limsup_{k\to\infty} \left(\|S''_k\| - r\right)_+ \;=\; 0\,.$$

It follows that the operator $\operatorname{diag}\{S''_1 - S_1, S''_2 - S_2, \ldots\}$ is compact and, consequently, $A = \operatorname{diag}\{S_1, S_2, \ldots\} + K$ where $K \in \mathcal{C}(H)$. □

Theorem 2.12 and Lemma 3.7 also imply the following quantitative version of the BDF theorem.

Theorem 3.8. *Let H be separable, and let $A \in \mathcal{B}(H)$ be an operator with $\|A\| \leqslant 1$ satisfying the condition* **(C)***.*

1. *If $[A^*, A] \in \mathcal{C}(H)$ then for each $\varepsilon > 0$ there exists a diagonal operator $T_\varepsilon \in \mathcal{B}(H)$ such that $A - T_\varepsilon \in \mathcal{C}(H)$, $\|T_\varepsilon\| \leqslant \|A\|$ and $\|A - T_\varepsilon\| \leqslant F\big(\|[A^*, A]\|\big) + \varepsilon$.*
2. *If $[A^*, A] \notin \mathcal{C}(H)$ then for each $\varepsilon > 0$ there exists a diagonal operator $T_\varepsilon \in \mathcal{B}(H)$ such that $\|A - T_\varepsilon\|_{\mathrm{ess}} \leqslant 2F\big(\|[A^*, A]\|_{\mathrm{ess}}\big)$ and*

$$\|A - T_\varepsilon\| \;\leqslant\; 5F\big(\|[A^*, A]\|\big) + 3F\Big(2F\big(\|[A^*, A]\|_{\mathrm{ess}}\big)\Big) + \varepsilon\,.$$

Proof. Since a block diagonal normal operator is represented by a diagonal matrix in the orthonormal basis formed by its eigenvectors, it is sufficient to construct a block diagonal normal T_ε satisfying the above conditions.

Assume first that $[A^*, A] \in \mathcal{C}(H)$. Then, by Corollary 2.10 and Lemma 3.7, we have $A = \operatorname{diag}\{S_1, S_2, \ldots\} + K$, where $K \in \mathcal{C}(H)$ and S_k are normal operators in finite dimensional subspaces H_k such that $\|S_k\| \leqslant \|A\|$. Let us denote by E_n the orthogonal projections onto the subspaces $\bigoplus_{k=1}^n H_k$ and define $\delta_n := \|K - E_n K E_n\|$. Since

$$[(E_nAE_n)^*, E_nAE_n] \;=\; E_n(A^*E_nA - AE_nA^*)E_n \;=\; E_n(A^*[E_n, A] + [A^*, A] + A[A^*, E_n])E_n\,,$$

$[E_n, A]E_n = [E_n, K]E_n = (E_nKE_n - K)E_n$ and $[A^*, E_n]E_n = (K^* - E_nK^*E_n)E_n$, we have

$$\|\,[(E_nAE_n)^*, E_nAE_n]\,\| \;\leqslant\; \|[A^*, A]\| + 2\delta_n\,\|A\| \;\leqslant\; \|[A^*, A]\| + 2\delta_n\,.$$

Applying (3.1) to the finite rank operators E_nAE_n, we can find normal operators A_n acting in E_nH such that $\|A_n\| \leqslant \|E_nAE_n\| \leqslant \|A\|$ and

$$\|E_nAE_n - A_n\| \leqslant F\big(\|[A^*,A]\| + 2\delta_n\big) + \delta_n\,.$$

The block diagonal operators $\tilde{T}_n := A_n \oplus \mathrm{diag}\{S_{n+1},\, S_{n+2},\, \ldots\}$ are normal, $\|\tilde{T}_n\| \leqslant \|A\|$ and

$$A - \tilde{T}_n \;=\; (E_nAE_n - A_n) + (K - E_nKE_n)\,, \qquad \forall n = 1,2,\ldots$$

The above identity implies that $A - \tilde{T}_n \in \mathcal{C}(H)$ and

$$\|A - \tilde{T}_n\| \;\leqslant\; F\big(\|[A^*,A]\| + 2\delta_n\big) + 2\delta_n\,, \qquad \forall n = 1,2,\ldots$$

Since K is compact, $\lim_{n\to\infty}\delta_n = 0$ and, consequently, $\lim_{n\to\infty}\|A - \tilde{T}_n\| = F\big(\|[A^*,A]\|\big)$. Thus we can take $T_\varepsilon := \tilde{T}_n$ with a sufficiently large n.

Assume now that $\|[A^*,A]\|_{\mathrm{ess}} > 0$. From (2.13) with $J = \mathcal{C}(H)$ it follows that

$$A \;=\; S + K + R\,,$$

where S is a bounded normal operator, $K \in \mathcal{C}(H)$ and R is a bounded operator with $\|R\| \leqslant 2\,F\big(\|[A^*,A]\|_{\mathrm{ess}}\big)$. Since $|F| \leqslant 1$, we have $\|S+K\| \leqslant 3$.

Let $A' := \frac{1}{3}\,(S+K)$. Then $[(A')^*,A'] \in \mathcal{C}(H)$, $\|A'\| \leqslant 1$ and, in view of Lemma 3.4, A' satisfies the condition **(C)**. Applying (1) to A', we can find a block diagonal normal operator T'_ε and a compact operators K'_ε such that $A' = T'_\varepsilon + K'_\varepsilon$ and $\|K'_\varepsilon\| \leqslant F\big(\|[(A')^*,A']\|\big) + \varepsilon/3$. The identities $3A' = S+K = A-R$ and the above estimates for $\|A\|$, $\|R\|$ and $\|S+K\|$ imply that

$$\|[(A')^*,A']\| \;\leqslant\; \frac{1}{9}\,\big(\|[A^*,A]\| + 2\,\|R\|\,\|S+K\| + 2\,\|A\|\,\|R\|\big) \;\leqslant\; \|[A^*,A]\| + 2F\big(\|[A^*,A]\|_{\mathrm{ess}}\big)\,.$$

Obviously, $A = 3T'_\varepsilon + 3K'_\varepsilon + R$ and $\|A - 3T'_\varepsilon\|_{\mathrm{ess}} = \|R\|_{\mathrm{ess}} \leqslant 2F\big(\|[A^*,A]\|_{\mathrm{ess}}\big)$. Since $\|[A^*,A]\|_{\mathrm{ess}} \leqslant \|[A^*,A]\|$ and the function F is nondecreasing and concave, from the above estimates it follows that

$$\|K'_\varepsilon\| \;\leqslant\; F\Big(\|[A^*,A]\| + 2F\big(\|[A^*,A]\|_{\mathrm{ess}}\big)\Big) + \frac{\varepsilon}{3} \;\leqslant\; F\big(\|[A^*,A]\|\big) + F\Big(2F\big(\|[A^*,A]\|_{\mathrm{ess}}\big)\Big) + \frac{\varepsilon}{3}$$

and, consequently,

$$\|A - 3T'_\varepsilon\| \;\leqslant\; \|R\| + 3\,\|K'_\varepsilon\| \;\leqslant\; 5F\big(\|[A^*,A]\|\big) + 3F\Big(2F\big(\|[A^*,A]\|_{\mathrm{ess}}\big)\Big) + \varepsilon\,.$$

Thus we can take $T_\varepsilon := 3T'_\varepsilon$. □

Remark 3.9. Since ε can be chosen arbitrarily small, the distance from an operator A satisfying the conditions of Theorem 3.8 to the set of diagonal operators does not exceed $5F\big(\|[A^*,A]\|\big) + 3F\Big(2F\big(\|[A^*,A]\|_{\mathrm{ess}}\big)\Big)$. If A is normal then this sum is equal to zero and Theorem 3.8 turns into the Weyl–von Neumann–Berg theorem for bounded operators.

3.4. **Truncations of normal operators.** Let G be a positive unbounded self-adjoint operator in a separable Hilbert space H whose spectrum consists of eigenvalues of finite multiplicity accumulating to ∞. Denote its spectral projections corresponding to the intervals $(0,\lambda)$ by P_λ, and let

$$N(\lambda) := \operatorname{rank} P_\lambda \quad \text{and} \quad N_1(\lambda) := \sup_{\mu\leqslant\lambda} (N(\mu) - N(\mu-1))\,.$$

If $B \in \mathcal{B}(H)$ and $[G,B] \in \mathcal{B}(H)$ then, according to [LS, Theorem 1.3],

$$\|(I-P_\lambda)BP_\lambda\|_2^2 \;\leqslant\; \|(I-P_\lambda)B(P_\lambda - P_{\lambda-1})\|_2^2 + \|(I-P_\lambda)[G,B](G-\lambda I)^{-1}P_{\lambda-1}\|_2^2\,. \tag{3.6}$$

A direct calculation shows that $\|(G-\lambda I)^{-1}P_{\lambda-1}\|_2^2 \leqslant \frac{\pi^2}{6}\, N_1(\lambda)$ (see [LS] for details). This estimate, (3.6) and the obvious inequality $\|P_\lambda - P_{\lambda-1}\|_2^2 \leqslant N_1(\lambda)$ imply that

$$\|(I-P_\lambda)BP_\lambda\|_2^2 \;\leqslant\; \left(\|B\|^2 + \frac{\pi^2}{6}\,\|[G,B]\|^2\right) N_1(\lambda)\,. \tag{3.7}$$

Let $A \in \mathcal{B}(H)$ be a normal operator such that $[G,A] \in \mathcal{B}(H)$, and let $A_\lambda := P_\lambda A P_\lambda$ be its truncation to the subspace $P_\lambda H$. Then $[A_\lambda^*, A_\lambda] = P_\lambda A^*(I-P_\lambda)AP_\lambda - P_\lambda A(I-P_\lambda)A^*P_\lambda$ and, consequently,

$$\|[A_\lambda^*, A_\lambda]\|_1 \;\leqslant\; \|P_\lambda A^*(I-P_\lambda)AP_\lambda\|_1 + \|P_\lambda A(I-P_\lambda)A^*P_\lambda\|_1\,.$$

Since $\|P_\lambda B^*(I-P_\lambda)BP_\lambda\|_1 = \operatorname{Tr}(P_\lambda B^*(I-P_\lambda)BP_\lambda) = \|(I-P_\lambda)BP_\lambda\|_2^2$, applying (3.7) with $B = A$ and $B = A^*$, we obtain

$$\|[A_\lambda^*, A_\lambda]\|_1 \;\leqslant\; C_A\, N_1(\lambda)\,, \tag{3.8}$$

where $C_A := 2\,\|A\|^2 + \frac{\pi^2}{3}\,\|[G,A]\|^2$. The inequalities (3.2) and (3.8) imply that

$$\inf_{T_\lambda} \|A_\lambda - T_\lambda\|_1 \;\leqslant\; N(\lambda)\, F\big(C_A\, N_1(\lambda)/N(\lambda)\big)\,, \tag{3.9}$$

where the infimum is taken over all normal operators T_λ acting in the finite dimensional subspace $P_\lambda H$.

Assume that there exist positive constants c and $\varkappa$ such that $N(\lambda) = c\lambda^\varkappa + o(\lambda^\varkappa)$ as $\lambda\to\infty$ (that is, we have a Weyl type asymptotic formula for the counting function $N(\lambda)$). Then $N_1(\lambda)/N(\lambda) \to 0$ and, consequently, $F\big(C_A\, N_1(\lambda)/N(\lambda)\big) \to 0$ as $\lambda\to\infty$. Therefore, in view of (3.9), there exist normal operators $\tilde T_\lambda$ acting in the subspaces $P_\lambda H$ such that $\|\lambda^{-\varkappa}A_\lambda - \tilde T_\lambda\|_1 \to 0$ as $\lambda\to\infty$. Roughly speaking, this means that, under the above conditions on A and $N(\lambda)$, the normalized truncations $\lambda^{-\varkappa}A_\lambda$ are asymptotically close to normal matrices with respect to the $\mathcal{S}_1$-norm.

Remark 3.10. The Weyl asymptotic formula holds for elliptic self-adjoint pseudodifferential operators on closed compact manifolds and differential operators on domains with appropriate boundary conditions. If G is a pseudodifferential operator of order 1 and A is the multiplication by a smooth function, as in the Szegö limit theorem [Sz], then A and $[G,A]$ are bounded in the corresponding space L_2 and we have (3.9).

Remark 3.11. In [LS], the classical Szegö limit theorem was extended to wide classes of self-adjoint (pseudo)differential operators G and A. More precisely, the authors proved that $\operatorname{Tr} f(A_\lambda) \sim \operatorname{Tr} P_\lambda f(A) P_\lambda$ as $\lambda \to \infty$ for all sufficiently smooth functions $f : \mathbb{R} \mapsto \mathbb{R}$ and all self-adjoint operators G and A satisfying the above conditions. If $f : \mathbb{C} \mapsto \mathbb{C}$ and the operator A is normal then the right hand side of the above asymptotic formula is well-defined. However, generally speaking, the truncations A_λ are not normal matrices and the left hand side does not make sense. The results of this subsection suggest that similar limit theorems can be obtained for (almost) normal operators A, provided that $\operatorname{Tr} f(A_\lambda)$ is understood in an appropriate sense. For instance, it is plausible that the asymptotic formula holds for all sufficiently smooth functions $f : \mathbb{C} \mapsto \mathbb{C}$ if one defines $\operatorname{Tr} f(A_\lambda) := \sum_j f(\mu_j)$, where μ_j are the eigenvalues of A_λ (see [Sa]).

Appendix A. Resolution of the identity

The proof of Theorem 2.1 is based on successive reductions of the operator $A \in L_{\mathrm{n}}$ to normal operators whose spectra do not contain certain subsets of the complex plane. One can think of this process as removing subsets from $\sigma(A)$. After each step we obtain a new normal operator lying in L_{n}. The main problem is that, in order to carry on the reduction procedure, one has to ensure that the removal of a subset Ω from the spectrum does not change the spectral projection corresponding to $\mathbb{C}\backslash\overline{\Omega}$, and that the new operator still satisfies the condition **(C)**. In our scheme this is guaranteed by the equality $A(I-\Pi_\Omega) = A_\Omega(I-\Pi_\Omega)$ and the condition (a$_4$).

Further on

- $\mathcal{D}_r(\lambda)$ is the open disc of radius r centred at $\lambda \in \mathbb{C}$, and $\partial\mathcal{D}_r(\lambda)$ is its boundary;
- $\mathbb{S} := \partial\mathcal{D}_1(0)$ is the unit circle about the origin.

We shall need the following lemmas which will be proved in the next two subsections.

Lemma A.1. *Let $A \in L_{\mathrm{n}}$, and let Π_Ω be the spectral projection of A corresponding to an open set $\Omega \subset \mathbb{C}$. If Ω is homeomorphic to the disc $\mathcal{D}_1(0)$ and $A - \mu I \in \overline{L_0^{-1}}$ for some $\mu \in \Omega$ then there exists a normal operator $R_\Omega : \Pi_\Omega H \mapsto \Pi_\Omega H$ such that*

(a$_1$) *$(A - R_\Omega)\Pi_\Omega \in L$ and, consequently, $A_\Omega := A(I - \Pi_\Omega) \oplus R_\Omega \in L_{\mathrm{n}}$;*
(a$_2$) *$\sigma(R_\Omega) \subset \partial\Omega$, so that $\sigma(A_\Omega) \subset (\sigma(A) \setminus \Omega) \cup \partial\Omega$;*
(a$_3$) *$A_\Omega - \lambda I \in L_0^{-1}$ for all $\lambda \in \Omega$;*
(a$_4$) *if A satisfies the condition* **(C)** *then so does the operator A_Ω.*

In other words, we can remove the set Ω from $\sigma(A)$ by adding a perturbation which does not change $A(I - \Pi_\Omega)$. Moreover,

(a$_5$) $\|A - A_\Omega\| = \|(A - R_\Omega)\Pi_\Omega\| \leqslant 2r$,

where r is the radius of the minimal disc containing Ω. This shows that the perturbation is small whenever Ω is a subset of a small disc. However, the new operator A_Ω may have additional spectrum lying on $\partial\Omega$.

In view of the above, Lemma A.1 is not sufficient for the study of operators with one dimensional spectra, as does not allow one to split the one dimensional spectrum into disjoint components. This problem is resolved by

Lemma A.2. *Let the conditions of Lemma* A.1 *be fulfilled. Assume, in addition, that*

(i) L *has real rank zero;*
(ii) $\sigma(A)\cap\Omega$ *is a subset of a simple contour* γ *which intersects* $\partial\Omega$ *at two points;*
(iii) $A-\mu I\in L_0^{-1}$ *for all* $\mu\in\Omega\setminus\gamma$.

Then there exists a normal operator $R_\Omega:\Pi_\Omega H\mapsto\Pi_\Omega H$ *satisfying the conditions* (a_1), (a_3), (a_4) *and*

(a_2') $\sigma(R_\Omega)\subset\gamma\cap\partial\Omega$, *so that* $\sigma(A_\Omega)\subset(\sigma(A)\setminus\Omega)\cup(\gamma\cap\partial\Omega)$

A.1. **Proof of Lemma A.1.** The proof proceeds in three steps.

A.1.1. Assume first that $\Omega=\mathcal{D}_\varepsilon(0)$ and $\mu=0$, that is, $A\in\overline{L_0^{-1}}$. For the sake of brevity, we shall denote $\Pi_\varepsilon:=\Pi_{\mathcal{D}_\varepsilon(0)}$, $R_\varepsilon:=R_{\mathcal{D}_\varepsilon(0)}$ and $A_\varepsilon:=A_{\mathcal{D}_\varepsilon(0)}$.

Let $A=V\,|A|$ be the polar decomposition of A. Let us consider a sequence of operators $B_n\in L_0^{-1}$ such that $B_n\to A$ as $n\to\infty$, and let $B_n=V_n\,|B_n|$ be their polar decompositions. Then $|B_n|\to|A|$ and, consequently, $V_n\,|A|\to V\,|A|$ as $n\to\infty$. Since $V_n\,|B_n|=|B_n^*|V_n$ and $|B_n^*|\to|A^*|=|A|$, we also have $|A|V_n\to|A|V$ as $n\to\infty$. It follows that $V_n\,\rho(|A|)\to V\rho(|A|)$ and $\rho(|A|)V_n\to\rho(|A|)V$ as $n\to\infty$ for every continuous function $\rho:\mathbb{R}_+\mapsto\mathbb{R}$ vanishing near the origin.

Let us fix continuous nonnegative functions ρ_1 and ρ_2 of $\mathbb{R}_+$ such that $\rho_1\equiv1$ and $\rho_2\equiv0$ on the interval $[\varepsilon,\infty)$, $\rho_1\equiv0$ near the origin, and $\rho_1^2+\rho_2^2\equiv1$. Let

$$S_n\;:=\;V\rho_1^2(|A|)\;+\;\rho_2(|A|)V_n\,\rho_2(|A|)\,.$$

The operators S_n belong to L because $V_n\in L$ (see Remark 1.1), $\rho(|A|)\in L$ for all continuous functions ρ, and $V\rho_1(|A|)=A\,\tilde\rho_1(|A|)$ where $\tilde\rho_1(\tau):=\tau^{-1}\rho_1(\tau)$ is a continuous function.

Since V commutes with $|A|$, we have

$$S_n-V_n\;=\;(V-V_n)\,\rho_1^2(|A|)-(V-V_n)\,(I-\rho_2(|A|))\,\rho_2(|A|)-(I-\rho_2(|A|))\,(V_n-V)\,\rho_2(|A|)\,.$$

By the above, the right hand side converges to zero as $n\to\infty$. Since $V_n\in L_0^{-1}\cap L_{\mathrm{u}}$ (see Lemma 1.7), this implies that $S_n\in L_0^{-1}$ for all sufficiently large n.

Let us fix n such that $S_n\in L_0^{-1}$ and consider the polar decomposition $S_n=U_n\,|S_n|$. By Lemma 1.7, $U_n\in L_0^{-1}\cap L_{\mathrm{u}}$. Since $S_n(I-\Pi_\varepsilon)=V(I-\Pi_\varepsilon)$, the operator S_n coincides with the orthogonal sum $V(I-\Pi_\varepsilon)\oplus S_n\Pi_\varepsilon$. The unitary operator U_n has the same block structure, $U_n=V(I-\Pi_\varepsilon)\oplus U_n\Pi_\varepsilon$.

Let R_ε be the restriction of $\varepsilon\,U_n$ to the subspace $\Pi_\varepsilon H$. Obviously, $\sigma(R_\varepsilon)\in\partial\mathcal{D}_\varepsilon(0)$. Since $(A-R_\varepsilon)\Pi_\varepsilon=A-f_\varepsilon(|A|)\,U_n$, where $f_\varepsilon(t):=\varepsilon+(t-\varepsilon)_+$ is a continuous function, the operator R_ε satisfies the condition (a_1).

We have $A_\varepsilon=A(I-\Pi_\varepsilon)\oplus R_\varepsilon=f_\varepsilon(|A|)\,U_n$, where $f_\varepsilon\geqslant\varepsilon>0$ and $U_n\in L_0^{-1}$. Therefore $A_\varepsilon\in L_0^{-1}$ (see Lemma 1.7). Since $\sigma(A_\varepsilon)\cap\mathcal{D}_\varepsilon(0)=\varnothing$, Lemma 1.5 implies that $A_\varepsilon-\lambda I\in\overline{L_0^{-1}}$ for all $\lambda\in\overline{\mathcal{D}_\varepsilon(0)}$.

It remains to prove that $A_\varepsilon-\lambda I\in\overline{L_0^{-1}}$ for $\lambda\not\in\overline{\mathcal{D}_\varepsilon(0)}$ whenever A satisfies the condition **(C)**. Let $\Pi_{\delta,\lambda}$ be the spectral projection of A corresponding to the open disc $\mathcal{D}_\delta(\lambda)$ of

radius $\delta < |\lambda| - \varepsilon$. Applying the above arguments to the operator $A - \lambda I$, we can find an operator $R_{\delta,\lambda}$ acting in $\Pi_{\delta,\lambda} H$ such that $\sigma(R_{\delta,\lambda}) \subset \partial\mathcal{D}_\delta(\lambda)$, $(A - R_{\delta,\lambda})\Pi_{\delta,\lambda} \in L$ and

$$A - (A - R_{\delta,\lambda})\Pi_{\delta,\lambda} - \lambda I \;=\; A(I - \Pi_{\delta,\lambda}) \oplus R_{\delta,\lambda} - \lambda I \;\in\; L_0^{-1}. \tag{A.1}$$

Denote

$$B_{t,\delta} \;:=\; ((1-t)A + tA_\varepsilon)\;(I - \Pi_{\delta,\lambda}) \oplus R_{\delta,\lambda}.$$

Since $\Pi_\varepsilon \Pi_{\delta,\lambda} = \Pi_{\delta,\lambda}\Pi_\varepsilon = 0$ and $A_\varepsilon = A - (A - R_\varepsilon)\Pi_\varepsilon$, we have

$$B_{t,\delta} \;=\; A(I - \Pi_{\delta,\lambda}) \oplus R_{\delta,\lambda} - t(A - R_\varepsilon)\Pi_\varepsilon \;=\; B_\delta^{(1)} \oplus B_t^{(2)} \oplus R_{\delta,\lambda}, \tag{A.2}$$

where $B_\delta^{(1)} := A(I - \Pi_{\delta,\lambda})(I - \Pi_\varepsilon)$ and $B_t^{(2)} := ((1-t)A + tR_\varepsilon)\,\Pi_\varepsilon$. Obviously, $\sigma(B_\delta^{(1)}) \subset \mathbb{C} \setminus \mathcal{D}_\delta(\lambda)$ and $\sigma(B_t^{(2)}) \subset \overline{\mathcal{D}_\varepsilon(0)}$ for all $t \in [0,1]$ (because $\|B_{t,\delta}^{(2)}\| \leqslant \varepsilon$). Thus the spectra of all the operators in the orthogonal sum on the right hand side of (A.2) do not contain the point λ. Therefore the operator $B_{t,\delta} - \lambda I$ is invertible. The first equality (A.2) implies that $B_{t,\delta} \in L$, so we have $B_{t,\delta} - \lambda I \in L^{-1}$ for all $t \in [0,1]$. By (A.1), $B_{0,\delta} - \lambda I \in L_0^{-1}$ and, consequently, $B_{1,\delta} - \lambda I \in L_0^{-1}$. Now, letting $\delta \to 0$, we obtain

$$\lim_{\delta \to 0} (B_{1,\delta} - \lambda I) \;=\; \lim_{\delta \to 0} (A_\varepsilon(I - \Pi_{\delta,\lambda}) \oplus R_{\delta,\lambda} - \lambda I) \;=\; A_\varepsilon - \lambda I \;\in\; \overline{L_0^{-1}}.$$

A.1.2. Let $B \in L_{\mathrm{n}}$, and let $\varphi : \mathbb{C} \to \mathbb{C}$ be a homeomorphism isotopic to the identity. The results obtained in Subsection A.1.1 imply that $\varphi(B)$ satisfies the condition **(C)** whenever so does the operator B.

Indeed, let us fix $\mu \in \mathbb{C}$ and consider the isomorphism $\psi : z \mapsto \varphi(z + \varphi^{-1}(\mu))$. Denote $A := B - \varphi^{-1}(\mu)I$, and let A_ε be the operators constructed in Subsection A.1.1. We have $\mu \notin \sigma(\psi(A_\varepsilon))$ for all $\varepsilon > 0$ because $\psi^{-1}(\mu) = 0 \notin \sigma(A_\varepsilon)$. Moreover, since φ is isotopic to the identity, the same is true for ψ and, by Lemma 1.5, $\psi(A_\varepsilon) - \mu I \in L_0^{-1}$ for all $\varepsilon > 0$. This implies that

$$\varphi(B) - \mu I \;=\; \psi(A) - \mu I \;=\; \lim_{\varepsilon \to 0} (\psi(A_\varepsilon) - \mu I) \;\in\; \overline{L_0^{-1}}.$$

A.1.3. Assume now that Ω is an arbitrary domain and $\mu \in \Omega$ is an arbitrary point satisfying the conditions of the lemma. Let us fix a homeomorphism $\psi : \mathbb{C} \mapsto \mathbb{C}$ isotopic to the identity such that $\psi : \Omega \mapsto \mathcal{D}_1(0)$ and $\psi(\mu) = 0$. Denote $\tilde{A} := \psi(A)$. Then Π_Ω coincides with the spectral projection of $\tilde{A}$ corresponding to the open disc $\mathcal{D}_1(0)$.

By A.1.1, there exists an operator $\tilde{R}_1$ acting in $\Pi_\Omega H$ such that $(\tilde{A} - \tilde{R}_1)\Pi_\Omega \in L$ and $\sigma(\tilde{R}_1) \subset \partial\mathcal{D}_1(0)$. Let $\tilde{A}_1 := \tilde{A}(I - \Pi_\Omega) \oplus \tilde{R}_1$ and $R_\Omega := \psi^{-1}(\tilde{R}_1)$. Obviously, the inverse image R_Ω satisfies (a_1) and (a_2), and $A_\Omega = \psi^{-1}(\tilde{A}_1)$. Since ψ is isotopic to the identity, Lemma 1.5 implies (a_3). Finally, by A.1.2, if A satisfies the condition **(C)** then the same is true for the operators $\tilde{A}$, $\tilde{A}_1$ (as was shown in A.1.1) and A_Ω. □

A.2. **Proof of Lemma A.2.** It is sufficient to prove the lemma in the case where $\Omega = \mathcal{D}_1(0)$ and $\gamma \cap \Omega = (-1,1)$. After that, the general result is obtained by choosing a homeomorphism ψ isotopic to the identity such that $\psi : \Omega \mapsto \mathcal{D}_1(0)$ and $\psi : \gamma \cap \Omega \mapsto (-1,1)$ and repeating the same arguments as in A.1.3.

Further on we always assume that Ω, γ and $\sigma(A)$ are as above and write Π_1, R_1 and A_1 instead of Π_Ω, R_Ω and A_Ω.

A.2.1. Suppose first that $\sigma(A)$ lies on a simple closed contour γ' homeomorphic to $\mathbb{S}$.

Let $\varphi : \mathbb{C} \mapsto \mathbb{C}$ be a homeomorphism isotopic to the identity such that $\varphi : \gamma' \mapsto \mathbb{S}$ and $\varphi(0) = -1$. The operator $\varphi(A)$ belongs to L_{u} because its spectrum lies in $\mathbb{S}$. The condition (iii) and Lemma 1.5 imply that $\varphi(A) \in L_0^{-1}$. Therefore, by Lemma 1.8, there exist operators $W_n \in L_{\mathrm{u}}$ such that $W_n \to \varphi(A)$ as $n \to \infty$ and $-1 \notin \sigma(W_n)$. Let $B_n := \varphi^{-1}(W_n)$ be their inverse images. Then B_n belong to L_{n}, $\sigma(B_n) \subset \gamma' \setminus \{0\}$ for all n, and $B_n \to A$ as $n \to \infty$.

The rest of this subsection is similar to Subsection A.1.1. Let us fix continuous nonnegative functions ρ_1 and ρ_2 of $\mathbb{R}_+$ such that $\rho_1 \equiv 1$ and $\rho_2 \equiv 0$ on the interval $[1, \infty)$, $\rho_1 \equiv 0$ near the origin, and $\rho_1^2 + \rho_2^2 \equiv 1$. Define

$$\tilde{S}_n \;:=\; V\rho_1^2(|A|) \;+\; \rho_2(|A|)\,(\operatorname{Re} V_n)\,\rho_2(|A|)$$

where V and V_n are the isometric operators in the polar representations $A = V|A|$ and $B_n = V_n|B_n|$. Note that

$$\rho_2(|B_n|)\,(\operatorname{Re} V_n)\,\rho_2(|B_n|) \;=\; (\operatorname{Re} V_n)\,\rho_2^2(|B_n|) \;=\; V_n\,\rho_2^2(|B_n|) \tag{A.3}$$

because $\sigma(B_n) \cap \mathcal{D}_1(0) \subset (-1, 1)$ and $\rho_2 \equiv 0$ outside the interval $[0, 1)$.

We have

$$\tilde{S}_n - V_n \;=\; (V - V_n)\,\rho_1^2(|A|) + \big(\rho_2(|A|)\,(\operatorname{Re} V_n)\,\rho_2(|A|) - V_n\,\rho_2^2(|A|)\big)\,.$$

Since $\rho_1 \equiv 0$ in a neighbourhood of the origin, the first term in the right hand side converges to zero. Since $\rho_2(|B_n|) \to \rho_2(|A|)$, the identity (A.3) implies that the second term also converges to zero. Thus $\|\tilde{S}_n - V_n\| \to 0$ as $n \to \infty$ and, consequently, $\tilde{S}_n \in L^{-1}$ for all sufficiently large n.

Let us fix n such that $\tilde{S}_n \in L^{-1}$ and consider the polar decomposition $\tilde{S}_n = \tilde{U}_n\,|\tilde{S}_n|$. The unitary operator $\tilde{U}_n$ has the same block structure as U_n in the proof of Lemma A.1 but now, in addition, its restrictions to the subspace $\Pi_1 H$ is self-adjoint. Let $R_1 = \tilde{U}_n\big|_{\Pi_1 H}$. Then R_1 satisfies (a$_1$) and its spectrum can contain only the points ± 1, so we have (a$_2'$) instead of (a$_2$). By Remark 1.6, A_1 satisfies the condition **(C)**, which implies (a$_3$) and (a$_4$).

A.2.2. Suppose now that $\sigma(A) \setminus (-1, 1)$ is an arbitrary subset of $\mathbb{C} \setminus \mathcal{D}_1(0)$.

In the process of proof we shall introduce auxiliary operators $A^{(1)}$ and $A^{(2)}$ lying in L_{n}, such that

($\star$) the spectral projection of $A^{(j)}$ corresponding to $\mathcal{D}_1(0)$ coincides with Π_1, and $A^{(j)}\,\Pi_1 = A\,\Pi_1$.

Every next operator will have a simpler spectrum, and R_1 will be defined in terms of $A^{(2)}$.

Let us consider the homotopy $\psi_t : \mathbb{C} \mapsto \mathbb{C}$ defined by

$$\psi_t(z) \;=\; \begin{cases} z & \text{if } z \in \mathcal{D}_1(0)\,, \\ (1-t)z + t\,\frac{z}{|z|} & \text{if } z \notin \mathcal{D}_1(0)\,, \end{cases} \qquad \text{where } t \in [0, 1]\,,$$

and let $A^{(1)} := \psi_1(A)$. Since $\psi_1(z) = z$ for all $z \in \mathcal{D}_1(0)$ and $\psi_1 : \mathbb{C} \setminus \mathcal{D}_1(0) \mapsto \mathbb{S}$, the operator $A^{(1)}$ satisfies the condition ($\star$) and $\sigma(A^{(1)}) \subset (-1, 1) \cup \mathbb{S}$. In view of Lemma 1.5,

$A^{(1)}$ also satisfies (iii). Denote by $\tilde\Pi$ the spectral projection of $A^{(1)}$ corresponding to the open lower semicircle $\mathbb{S}_- := \{z\in\mathbb{S} : \operatorname{Im} z<0\}$.

Now let us consider the homotopy $\varphi_t : \overline{\mathcal{D}_1(0)} \mapsto \overline{\mathcal{D}_1(0)}$ defined by

$$\varphi_t \;=\; \begin{cases} z - it\,\operatorname{Im} z + it\sqrt{1-(\operatorname{Re} z)^2} & \text{if } \operatorname{Im} z \geqslant 0\,,\\ z + it\sqrt{1-(\operatorname{Re} z)^2} & \text{if } \operatorname{Im} z\leqslant 0\,,\end{cases} \qquad \text{where } t\in[0,1]\,,$$

and let $\tilde A := \varphi_1(A^{(1)})$. Since $\varphi_1 : \mathbb{S}_- \mapsto (-1,1)$, $\varphi_1 : (-1,1)\mapsto \mathbb{S}_+$ and $\varphi_1:\mathbb{S}_+\mapsto\mathbb{S}_+$, the spectrum $\sigma(\tilde A)$ lies on the contour γ' formed by the interval $[-1,1]$ and the upper semicircle $\mathbb{S}_+ := \{z\in\mathbb{S} : \operatorname{Im} z>0\}$. By Lemma 1.5, the operator $\tilde A$ satisfies (iii).

Note that $\varphi_1|_{\mathbb{S}_-}$ is a homeomorphism between $\mathbb{S}_-$ and $(-1,1)$. Therefore, the spectral projection of $\tilde A$ corresponding to the interval $(-1,1)$ coincides with $\tilde\Pi$. Applying A.2.1 to the operator $\tilde A$, we can find a self-adjoint operator $\tilde R$ acting in the subspace $\tilde\Pi H$ such that $(\tilde A-\tilde R)\,\tilde\Pi\in L$, $\sigma(\tilde R)\subset\{-1,1\}$ and $\tilde A\,(I-\tilde\Pi)\oplus\tilde R$ satisfies the condition (iii). Since the restriction of $\tilde A$ to $\tilde\Pi H$ is self-adjoint, we have $(\tilde A-\tilde R)\,\tilde\Pi\in L_{\mathrm{s}}$.

Let $A^{(2)} := A^{(1)} - \tilde\varphi_1^{-1}((\tilde A-\tilde R)\,\tilde\Pi)$, where $\tilde\varphi_1^{-1} : (-1,1)\mapsto\mathbb{S}_-$ is the inverse mapping $\left(\varphi_1|_{\mathbb{S}_-}\right)^{-1}$. Then $A^{(2)} = A^{(1)}(I-\tilde\Pi)\oplus\tilde R$. This implies that $\sigma(A^{(2)})\subset\gamma'$ and $A^{(2)}$ satisfies the condition ($\star$). Moreover, $A^{(2)}$ satisfies (iii) because

$$[0,1]\ni t \;\mapsto\; A^{(1)} - t\tilde\varphi_1^{-1}((\tilde A-\tilde R)\,\tilde\Pi) - \mu I$$

is a path in L^{-1} from $A^{(1)}-\mu I$ to $A^{(2)}-\mu I$ for each μ lying in the open domain bounded by γ'.

Finally, applying A.2.1 to $A^{(2)}$, we obtain an operator R_1 acting in the subspace $\Pi_1 H$ such that $(A^{(2)}-R_1)\,\Pi_1 = (A-R_1)\,\Pi_1\in L_{\mathrm{s}}$ and $\sigma(R_1)\subset\{-1,1\}$. The latter inclusion and (ii) imply that $\sigma((A-tA+tR_1)\,\Pi_1)\subset[-1,1]$ for all $t\in[0,1]$. Thus we have

$$A - t(A-R_1)\,\Pi_1 - \mu I \;\in\; L^{-1}\,,\qquad \forall\mu\in\mathcal{D}_1(0)\setminus(-1,1)\,,\quad \forall t\in[0,1]\,.$$

Since the operator A satisfies (iii), it follows that $A_1-\mu I\in L_0^{-1}$ for all $\mu\in\mathcal{D}_1(0)$, where $A_1 = A(I-\Pi_1)\oplus R_1$. Now (a$_3$) and (a$_4$) are proved in the same way as in Lemma A.1. $\square$

A.3. **Proof of Theorem 2.1.** Every open set Ω_j coincides with the union of a collection of open discs. Since the spectrum $\sigma(A)$ is compact, it is sufficient to prove the theorem assuming that Ω_j is the union of a finite collection of open disks $D_{j,k}$. If there exist mutually orthogonal projections $P_{j,k}$ such that $\sum_{j,k}P_{j,k}=I$ and $P_{j,k}H\subset\Pi_{D_{j,k}}H$ then we can take $P_j := \sum_k P_{j,k}$. Thus we only need to prove the theorem for open discs Ω_j. In the rest of the proof we shall be assuming that $\Omega_j = \mathcal{D}_{r_j}(z_j)$.

The proof is by induction on m. If $m=1$ then the result is obvious. Suppose that the theorem holds for $m-1$ and consider a family of m open discs $\{\Omega_j\}_{j=1}^m$ covering $\sigma(A)$. If $\Omega_k\subset\bigcup_{j\neq k}\Omega_j$ for some k then we can take $P_k=0$ and apply the induction assumption. Further on we shall be assuming that $\Omega_k\not\subset\bigcup_{j\neq k}\Omega_j$ for all $k=1,\dots,m$.

A.3.1. If $r > t > 0$, let

$$\mathcal{D}_r := \mathcal{D}_r(z_m) \quad \text{and} \quad \mathcal{D}_{t,r} := \{z \in \mathbb{C} \,:\, t < |z - z_m| < r\}\,.$$

Note that

$$\sigma(A) \setminus \mathcal{D}_t \;\subset\; \bigcup_{j=1}^{m-1} \Omega_j \tag{A.4}$$

whenever $r_m - t$ is small enough. Indeed, if this were not true then there would exist a sequence of points $\mu_n \in \sigma(A) \setminus \left(\bigcup_{j=1}^{m-1} \Omega_j\right)$ converging to $\partial\Omega_m$, and the limit point would not belong to $\bigcup_{j=1}^{m} \Omega_j$.

In the rest of the proof $t \in (0, r_m)$ is assumed to be so close to r_m that (A.4) holds.

A.3.2. In this subsection we are going to construct auxiliary operators $A^{(i)} \in L_{\mathrm{n}}$ satisfying **(C)** and the following condition

($\star\star$) $\Pi^{(i)}_{\Omega_j} H \subset \Pi_{\Omega_j} H$ for all $j = 1, \dots, m$, where $\Pi^{(i)}_{\Omega_j}$ are the spectral projections of $A^{(i)}$ corresponding to Ω_j.

Assume first that $\partial\Omega_m \cap \Omega_{j_k} \neq \varnothing$ for some indices $j_k \leqslant m-1$. Let us fix an arbitrary $r \in (t, r_m)$ and consider the open annulus $\mathcal{D}_{t,r}$. The circles $\partial\Omega_{j_k}$ split $\mathcal{D}_{t,r}$ into a finite collection of connected disjoint open sets Λ_α such that $\overline{\mathcal{D}_{t,r}} = \bigcup_\alpha \overline{\Lambda_\alpha}$. Each set Λ_α is a circular polygon whose edges are arcs of the circles $\partial\Omega_{j_k}$, $\partial\mathcal{D}_t$ and $\partial\mathcal{D}_r$. Since $\Omega_k \not\subset \mathcal{D}_{t,r}$ for all $k = 1, \dots, m$, the boundaries $\partial\Lambda_\alpha$ are connected and, consequently, each polygon Λ_α is homeomorphic to a disc.

Let us remove from $\sigma(A)$ the open sets Λ_α, repeatedly applying Lemma A.1. Then we obtain an operator $A^{(1)} \in L_{\mathrm{n}}$ satisfying the condition **(C)**, such that

$$\sigma(A^{(1)}) \;\subset\; \overline{\mathcal{D}_t} \cup (\cup_\alpha \partial\Lambda_\alpha) \cup (\mathbb{C} \setminus \mathcal{D}_r) \quad \text{and} \quad \sigma(A^{(1)}) \setminus \overline{\mathcal{D}_r} = \sigma(A) \setminus \overline{\mathcal{D}_r}\,.$$

Note that $\Lambda_\alpha \subset \Omega_j$ whenever $\partial\Lambda_\alpha \cap \Omega_j \neq \varnothing$. In view of ($\mathrm{a}_2$), this implies that the removal of Λ_α from the spectrum can only reduce the eigenspace corresponding to Ω_j. Therefore $A^{(1)}$ satisfies the condition ($\star\star$).

Now, repeatedly applying Lemma A.2, let us remove from $\sigma(A^{(1)}) \cap \mathcal{D}_{t,r}$ the interiors of all edges of the polygons $\partial\Lambda_\alpha$ lying in the open annulus $\mathcal{D}_{t,r}$. Then we obtain an operator $A^{(2)} \in L_{\mathrm{n}}$ satisfying the condition **(C)**, such that

$$\sigma(A^{(2)}) \;\subset\; \overline{\mathcal{D}_t} \cup \Sigma \cup \partial\mathcal{D}_r \cup \left(\sigma(A) \setminus \overline{\mathcal{D}_r}\right) \quad \text{and} \quad \sigma(A^{(2)}) \setminus \overline{\mathcal{D}_r} = \sigma(A) \setminus \overline{\mathcal{D}_r}\,, \tag{A.5}$$

where Σ is the set of vertices of the polygons Λ_α. If at least one point of a closed edge of Λ_α belongs to Ω_j, then the interior part of this edge also lies in Ω_j. In view of (a_2'), this implies that the removal of open arcs does not increase the eigenspaces corresponding to Ω_j. Therefore $A^{(2)}$ satisfies ($\star\star$).

By (A.5), the set of points $z \in \sigma(A^{(2)}) \setminus \mathcal{D}_r$ which do not belong to $\sigma(A) \setminus \mathcal{D}_r$ consists of a countable collection of arcs γ_β of the circle $\partial\mathcal{D}_r$, whose end points belong either to $\Sigma \bigcap \partial\mathcal{D}_r$ or to $\sigma(A) \bigcap \partial\mathcal{D}_r$. Each interior point of γ_β is separated from $\sigma(A^{(2)}) \setminus \overline{\mathcal{D}_r}$ (otherwise it would belong to $\sigma(A)$). The set Σ is finite and, by (A.4), the intersection $\sigma(A) \bigcap \partial\mathcal{D}_r$ is a subset of $\bigcup_{j=1}^{m-1} \Omega_j$. This implies that $\left(\sigma(A^{(2)}) \bigcap \partial\mathcal{D}_r\right) \setminus \left(\bigcup_{j=1}^{m-1} \Omega_j\right)$ is covered by a finite

subcollection of arcs $\gamma_{\beta'}$ whose end points belong to $\Sigma\bigcup\left(\bigcup_{j=1}^{m-1}\Omega_j\right)$. Repeatedly applying Lemma A.2, let us remove the interior parts of the arcs $\gamma_{\beta'}$ from $\sigma(A^{(2)})$. Then we obtain an operator $A^{(3)}\in L_{\mathrm{n}}$ satisfying the condition **(C)**, such that

$$\sigma(A^{(3)})\ \subset\ \overline{\mathcal{D}_t}\cup\Sigma\cup\left(\cup_{j=1}^{m-1}\Omega_j\setminus\mathcal{D}_r\right)\,. \tag{A.6}$$

For the same reason as before, $A^{(3)}$ also satisfies the condition $(\star\star)$.

If $\partial\Omega_m\cap\Omega_j=\varnothing$ for all $j=1,\dots,m-1$ then $\sigma(A)$ is separated from the boundary $\partial\Omega_m$, and we define $A^{(3)}=A$. Obviously, in this case $A^{(3)}$ also satisfies $(\star\star)$ and (A.6) with $\Sigma=\varnothing$ and some $t\in(0,r_m)$ and $r\in(t,r_m)$.

A.3.3. Let P_m be the spectral projection of the operator $A^{(3)}$ corresponding to the set $\overline{\mathcal{D}_t}\cup\Sigma$. Since $t<r$ and Σ is finite, this set is separated from $\sigma(A^{(3)})\setminus\mathcal{D}_r$ and, consequently, $P_m\in L$. Since $\overline{\mathcal{D}_t}\cup\Sigma\subset\Omega_m$, the condition $(\star\star)$ implies that $P_mH\subset\Pi_{\Omega_m}H$.

Given $z\in\mathbb{C}$, let us consider the operator

$$A_z\ :=\ z\,P_m+(I-P_m)\,A^{(3)}\,.$$

From (A.6) it follows that

$$\sigma(A_z)\ \subset\ \{z\}\cup\left(\cup_{j=1}^{m-1}\Omega_j\setminus\mathcal{D}_r\right),\qquad\forall z\in\mathbb{C}\,. \tag{A.7}$$

If $z\in\mathcal{D}_t$ then for each sufficiently small $\delta>0$ there is a homeomorphism $\varphi_{z,\delta}:\mathbb{C}\mapsto\mathbb{C}$ isotopic to the identity, which maps a neighbourhood of $\overline{\mathcal{D}_t}\cup\Sigma$ onto $\mathcal{D}_\delta(z)$ and coincides with the identity on a neighbourhood of $\sigma(A^{(3)})\setminus\mathcal{D}_r$. By A.1.2, all the operators $\varphi_{z,\delta}(A^{(3)})$ satisfy the condition **(C)**. Since $\varphi_{z,\delta}(A^{(3)})\to A_z$ as $\delta\to0$, this implies that A_z also satisfy the condition **(C)** for all $z\in\mathcal{D}_t$.

If $z\notin\mathcal{D}_t$ and $\lambda\neq z$, let us fix a point $\tilde z\in\mathcal{D}_t$ and a path $\mu(s)$ from $\tilde z$ to z which does not go through λ. Assume that $\varepsilon>0$ is so small that $\tilde z\notin\mathcal{D}_\varepsilon(\lambda)$. Then, applying Lemma A.1 with $\Omega=\mathcal{D}_\varepsilon(\lambda)$ to $A_{\tilde z}$, we can find an operator $A_{\tilde z,\varepsilon}:=A_{\tilde z,\Omega}\in L_{\mathrm{n}}$ such that $P_mA_{\tilde z,\varepsilon}=A_{\tilde z,\varepsilon}P_m=\tilde zP_m$, $A_{\tilde z,\varepsilon}-\lambda I\in L_0^{-1}$ and $\lim_{\varepsilon\to0}A_{\tilde z,\varepsilon}=A_{\tilde z}$. Since $\mu(s)P_m+A_{\tilde z,\varepsilon}(I-P_m)-\lambda I$ is a path in L^{-1} from $A_{\tilde z,\varepsilon}-\lambda I$ to $zP_m+A_{\tilde z,\varepsilon}(I-P_m)-\lambda I$, the latter operator also belongs to L_0^{-1}. Therefore

$$A_z-\lambda I\ =\ \lim_{\varepsilon\to0}(zP_m+A_{\tilde z,\varepsilon}(I-P_m)-\lambda I)\ \in\ \overline{L_0^{-1}}\,,\qquad\forall\lambda\neq z\,.$$

Obviously, the same inclusion holds for $\lambda=z$. Thus the operators A_z satisfy the condition **(C)** for all $z\in\mathbb{C}$.

A.3.4. Let us fix an arbitrary point $z'\in\Omega_1\setminus\left(\bigcup_{j=2}^{m}\Omega_j\right)$ and denote $A':=A_{z'}$. In view of (A.7), we have $\sigma(A')\subset\bigcup_{j=1}^{m-1}\Omega_j$. Applying the induction assumption to the operator A', we can find mutually orthogonal projections $P_1',P_2,\dots,P_{m-1}$ such that $P_1'+\sum_{j=2}^{m-1}P_j=I$, $P_1'H\subset\Pi_{\Omega_1}'H$ and $P_jH\subset\Pi_{\Omega_j}'H$ for all $j=2,\dots,m-1$, where Π_{Ω_j}' are the spectral projections of A' corresponding to Ω_j. Since $z'\notin\bigcup_{j=2}^{m-1}\Omega_j$, the projections $\Pi_{\Omega_2}',\dots,\Pi_{\Omega_{m-1}}'$ coincide with the spectral projections of the truncation $A^{(3)}\big|_{(I-P_m)H}$. Thus we have $P_m\Pi_{\Omega_j}'=0$ for all $j=2,\dots,m-1$ and, consequently, $P_mH\subset P_1'H$.

Let $P_1 := P_1' - P_m$. Then, by the above, $P_1, \dots, P_m$ are mutually orthogonal projections such that $\sum_{j=1}^m P_j = I$. It remains to notice that, in view of ($\star\star$),

$$(P_1' - P_m)\,H \;\subset\; \Pi^{(3)}_{\Omega_1} H \;\subset\; \Pi_{\Omega_1} H$$

and $\Pi'_{\Omega_j} H \subset \Pi^{(3)}_{\Omega_j} H \subset \Pi_{\Omega_j} H$ for all $j = 2, \dots, m-1$. □

Appendix B. Remarks and references

B.1. One can easily show that $L_{\rm f} \cap L_{\rm s} \subset \overline{L^{-1} \cap L_{\rm s}}$ and $L_{\rm f} \cap L_{\rm n} \subset \overline{L_0^{-1} \cap L_{\rm n}}$ in any C^*-algebra L (see the proof of Corollaries 2.4 and 2.5). If L has real rank zero then

(1) $L_{\rm s} = \overline{L_{\rm f} \cap L_{\rm s}}$ (this is the implication $(1) \Rightarrow (3)$ in Corollary 2.4) and
(2) $L_0^{-1} \cap L_{\rm u} = \overline{L_{\rm f} \cap L_{\rm u}}$.

The first result is well known and elementary (see, for example, [D, Theorem V.7.3] or [BP, Theorem 2.6]). The second is due to Huaxin Lin [L1]. Note that (2) is an immediate consequence of (1) and Lemma 1.8.

Using Lemma 2.3, one can deduce from Theorem 2.1 "quantitative" versions of (1) and (2), where the distance to an approximating operator with a finite spectrum σ is estimated in terms of σ.

B.2. Theorem 2.1 remains valid for self-adjoint operators A in a general C^*-algebra L satisfying the condition

$(\mathbf{C}_{\rm s})$ $A - \lambda I \in \overline{L^{-1} \cap L_{\rm s}}$ for all $\lambda \in \mathbb{R}$.

Indeed, if we take $B_n \in L^{-1} \cap L_{\rm s}$ in Subsection A.1.1 then the operators V_n, S_n and U_n are self-adjoint, and so is the operator A_ε. The same arguments show that A_ε still satisfies the condition $(\mathbf{C}_{\rm s})$. Therefore, iterating this procedure, we can remove from $\sigma(A)$ an arbitrary finite collection of open intervals without changing the spectral projections corresponding to the complements of their closures. This allows us to construct approximate spectral projections in the same manner as in Subsection A.3, with obvious simplifications due to the fact that $\sigma(A) \subset \mathbb{R}$.

Using this observation, one can refine Corollary 2.4 as follows.

B.3. *In an arbitrary C^*-algebra L, the following statements about a self-adjoint operator $A \in L_{\rm s}$ are equivalent.*

(1) *The operator A satisfies the condition $(\mathbf{C}_{\rm s})$.*
(2) *The operator A has approximate spectral projections in the sense of Theorem 2.1, associated with any finite open cover of its spectrum.*
(3) $A \in \overline{L_{\rm f} \cap L_{\rm s}}$.

As explained in Subsection B.2, (2) follows from (1), and the other two implications are proved in the same way as in Subsection 2.1.

B.4. It is clear from the proof that Lemma A.1 remains valid if we replace L_0^{-1} with L^{-1}. However, in Lemma A.2 the assumption (iii) is of crucial importance.

B.5. For a disc $\Omega = \mathcal{D}_\varepsilon(0)$, Lemma A.1 without the condition (a_4) can easily be deduced from [P, Theorem 5] (see also [R, Theorem 2.2]). In the both papers the theorem was proved for $A \in \overline{L^{-1}}$, but in [FR2, Section 3] the authors explained that the approximating operator belongs $\overline{L_0^{-1}}$ whenever $A \in \overline{L_0^{-1}}$.

[P, Theorem 5] holds if $\operatorname{dist}(A, L^{-1}) < \varepsilon$, whereas we assumed that $\operatorname{dist}(A, L_0^{-1}) = 0$ and, in addition, that A is normal. Our proof slightly differs from those in [P], [R] and [FR2]. It gives a weaker result in the general case but is better suited for the study of operators with one dimensional spectra. It also shows that one can choose approximating operators satisfying the condition **(C)**.

B.6. Lemma A.2 seems to be new. Possibly, one could deduce the (1) $\Rightarrow$ (3) part of Corollary 2.5 from [L2, Theorem 5.4], but our approach gives more information about the approximating operators. In particular, Theorem 2.1 implies a quantitative (in the same sense as in Subsection B.1) version of [L2, Theorem 5.4].

B.7. One can further refine Theorem 2.12 by introducing subsets $M_T^n \subset M_T$, which consist of convex combinations of operators of the form $S_1 T S_2$ containing at most n terms. The same proof shows that $M_{[A^*,A]}$ in (2.5) can be replaced with $M_{[A^*,A]}^{n(\varepsilon)}$ where $n(\varepsilon)$ is an integer-valued nonincreasing function of $\varepsilon \in (0,\infty)$.

B.8. A review of results on almost commuting operators and matrices can be found in [DS]. The authors listed several dimension-dependent results and discussed the following known example.

Let A_m and B_m be $(m+1)\times(m+1)$-matrices defined by the identities

$$A_m e_j = \left(1 - \tfrac{2j}{m}\right) e_j \text{ for all } j = 0,\dots,m,$$
$$B_m e_j = \tfrac{2}{m+1}\sqrt{(j+1)(m-j)}\, e_{j+1} \text{ for all } j = 0,\dots,m-1, \text{ and } B_m e_m = 0,$$

where $\{e_0, e_1, \dots, e_m\}$ is an orthonormal basis in $\mathbb{C}^{m+1}$. Then $\|A_m\| = 1$, $\|B_m\| \leqslant 1$, $A_m = A_m^*$, $\|[B_m^*, B_m]\| \leqslant \frac{4}{m}$ and $\|[A_m, B_m]\| \leqslant \frac{2}{m}$, so that the Hermitian matrices A_m, $\operatorname{Re} B_m$ and $\operatorname{Im} B_m$ are almost commuting for large values of m. However, the distance between the pair $\{A_m, B_m\}$ and any pair of commuting $(m+1)\times(m+1)$-matrices is estimated from below by a constant independent of m [Ch] (see also [V]).

This example shows that, without additional assumptions, $\mathrm{B}(\varepsilon)$ in (2.5) cannot be replaced by $\mathrm{B}(\varepsilon)\cap L_{\mathrm{s}}$ (or, in other words, it is not sufficient to adjust only one operator in a pair of almost commuting self-adjoint operators to obtain a pair of commuting self-adjoint operators). Indeed, if (2.5) held with $\mathrm{B}(\varepsilon)\cap L_{\mathrm{s}}$ then, applying Theorem 2.12 to the matrices $\operatorname{Re} B_m + iA_m$ and $\operatorname{Im} B_m + iA_m$, we could find Hermitian $(m+1)\times(m+1)$-matrices X_m and Y_m such that $[A_m, X_m] = [A_m, Y_m] = 0$ and $\|X_m + iY_m - B_m\| \to 0$ as $m \to \infty$.

B.9. Theorem 2.12 allows one to obtain approximation results for operators $A \in \mathcal{B}(H)$ with compact self-commutators. For instance, if $A \in \mathcal{B}(H)$ satisfies the condition **(C)**, $\|A\| \leqslant 1$, $[A^*, A] \in \mathcal{S}_p$ and $\|[A^*, A]\|_p \leqslant c$ then the number of eigenvalues of each operator from $h(\varepsilon) M_{[A^*,A]}$ lying outside the interval $(-\varepsilon, \varepsilon)$ does not exceed $(c\,\varepsilon^{-1} h(\varepsilon))^p$. In view of (2.5), this implies that for each $\varepsilon > 0$ there exist a normal operator T_ε and a self-adjoint

operator R_ε of finite rank such that $\|A - T_\varepsilon - R_\varepsilon\| \leqslant 2\varepsilon$ and $\operatorname{rank} R_\varepsilon \leqslant (c\,\varepsilon^{-1}h(\varepsilon))^p$. Moreover, if the operator A is compact then one can take $T_\varepsilon \in \mathcal{C}(H)$.

Since Theorem 2.12 does not give an explicit estimate for $h(\varepsilon)$, the above observation is of limited interest. However, it shows that $\operatorname{rank} R_\varepsilon$ is bounded by a constant depending only on ε and p.

B.10. If $A \in B(H)$ and $\varepsilon > 0$, let us define

$$\operatorname{Spec}_\varepsilon(A) \;:=\; \sigma(A) \bigcup \left\{ z \in \mathbb{C} \,:\, \|(A - zI)^{-1}\| > \varepsilon^{-1} \right\}.$$

The set $\operatorname{Spec}_\varepsilon(A)$ is called the *ε-pseudospectrum* of A. It is known that

$$\operatorname{Spec}_\varepsilon(A) = \bigcup_{\|R\|<\varepsilon} \sigma(A+R) \quad \text{and} \quad \bigcap_{\delta>\varepsilon} \operatorname{Spec}_\delta(A) = \overline{\operatorname{Spec}_\varepsilon(A)}$$

(see, for instance, [Da, Theorem 9.2.13]) and [CCH, Lemma 2]). Let

$$d_A(\varepsilon) \;:=\; \sup_{\lambda \in \operatorname{Spec}_\varepsilon(A)} \operatorname{dist}(\lambda, \sigma_{\mathrm{ess}}(A)) \quad \text{and} \quad d_A(0) \;:=\; \sup_{\lambda \in \sigma(A)} \operatorname{dist}(\lambda, \sigma_{\mathrm{ess}}(A))\;,$$

where $\sigma_{\mathrm{ess}}(A)$ is the spectrum of the corresponding element of the Calkin algebra.

In [BD] the authors proved the following statement. If $\|[A^*, A]\| \leqslant c^2$ and

$$\|(A - \lambda I)^{-1}\| \;\leqslant\; (\operatorname{dist}(\lambda, \sigma_{\mathrm{ess}}(A)) - c)^{-1}\,, \qquad \forall \lambda : \operatorname{dist}(\lambda, \sigma_{\mathrm{ess}}(A)) > c\,,$$

then the normal operator T in the BDF theorem can be chosen in such a way that $\sigma(T) = \sigma_{\mathrm{ess}}(A)$ and $\|A - T\| \leqslant f(c)$, where $f : [0,\infty) \to [0,\infty)$ is some (unknown) continuous function vanishing at the origin that depends only on $\sigma_{\mathrm{ess}}(A)$.

Note that, under the above condition on $(A - \lambda I)^{-1}$, we have $d_A(\varepsilon) \leqslant c + \varepsilon$ for all $\varepsilon > 0$. Theorem 3.8(1) implies the following more precise result which holds without any a priori assumptions about the resolvent.

B.11. *Under the conditions of the BDF theorem, there exists a normal operator T such that $\sigma(T) = \sigma_{\mathrm{ess}}(A)$, $A - T \in \mathcal{C}(H)$ and*

$$\|A - T\| \;\leqslant\; 2\,\|A\|\, F\big(\|A\|^{-2}\|[A^*, A]\|\big) \;+\; d_A\left(2\,\|A\|\, F\big(\|A\|^{-2}\|[A^*, A]\|\big)\right)\,,$$

where $F : [0,\infty) \mapsto [0,1]$ is a nondecreasing concave function vanishing at the origin, which does not depend on A.

Indeed, applying Theorem 3.8(1) to the operator $\|A\|^{-1}A$, we can find a normal operator T' such that $A - T' \in \mathcal{C}(H)$ and $\|A - T'\| \leqslant 2\,\|A\|\, F\big(\|A\|^{-2}\|[A^*, A]\|\big)$. By the above, $\sigma(T') \subset \operatorname{Spec}_\delta(A)$ for all $\delta > \|A - T'\|$ and, consequently,

$$\operatorname{dist}(\lambda, \sigma_{\mathrm{ess}}(T')) \;=\; \operatorname{dist}(\lambda, \sigma_{\mathrm{ess}}(A)) \;\leqslant\; d_A\big(\|A - T'\|\big)\,, \qquad \forall \lambda \in \sigma(T')\,.$$

Now, using the spectral theorem, one can easily construct a normal operator T such that $T - T' \in \mathcal{C}(H)$, $\sigma(T) = \sigma_{\mathrm{ess}}(T')$ and $\|T - T'\| \leqslant d_A\big(\|A - T'\|\big)$. This operator satisfies the required conditions.

B.12. Theorem 2.12 states that (2.5) holds for all C^*-algebras L of real rank zero with some universal function h. The function F is determined only by h and, therefore, (2.11) is true for all C^*-algebras L of real rank zero and all seminorms satisfying the conditions (2.10). Our proof is by contradiction and does not give explicit estimates for h and F.

For a particular C^*-algebra L and a seminorm $\|\cdot\|_*$ on L, it may be possible to optimize the choice of functions h and F or to obtain additional information about their behaviour. Note that

(i) if (2.5) holds with some function h then we also have (2.11) with F defined by (2.9) for all seminorms $\|\cdot\|_*$ satisfying (2.10);
(ii) $\liminf_{\varepsilon\to 0}(\varepsilon\, h(\varepsilon)) > 0$ for any function h satisfying (2.5) and
(iii) $\liminf_{t\to 0}\big(t^{-1/2}F(t)\big) > 0$ for any function F satisfying (2.11)

(otherwise we obtain a contradiction by substituting an operator δA and letting $\delta \to 0$).

B.13. In [DS] the authors conjectured that the estimate (3.1) holds with a function F such that $F(t) \sim t^{1/2}$ as $t \to 0$. In the recent paper [Ha], Hastings proved (3.1) with $F(t) = t^{1/6}\tilde F(t)$, where $\tilde F$ is a function growing slower than any power of t as $t \to 0$.

Since the proof of Theorem 3.8(1) uses only (3.1), Hastings' result implies the following corollary.

B.14. *Let A satisfy the conditions of the BDF theorem, and let $\|A\| \leqslant 1$. Then for each $\varepsilon, \delta > 0$ there exists a diagonal operator $T_{\varepsilon,\delta}$ such that $A - T_{\varepsilon,\delta} \in \mathcal{C}(H)$ and*

$$\|A - T_{\varepsilon,\delta}\| \;\leqslant\; C_\delta\, \|[A^*, A]\|^{1/6-\delta} \;+\; \varepsilon\,,$$

where C_δ is a constant depending only on δ.

B.15. In most statements, for the sake of simplicity, we assumed that $\|A\| \leqslant 1$. One can easily get rid of this condition by applying the corresponding result to the operator $\|A\|^{-1}A$ (as was done in Subsection B.11).

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